Chaos & Fractals - Solutions to writing assignment 1

Let $t : [0, 1) \to [0, 1)$ denote the tripling map

$$t(x) = 3x \mod 1 = \begin{cases} 3x & 0 \leq x \leq 1/3 \\ 3x - 1 & 1/3 \leq x \leq 2/3 \\ 3x - 2 & 2/3 \leq x < 1. \end{cases}$$

and let $T : [0, 1] \to [0, 1]$ denote the tall tent map

$$T(x) = \begin{cases} 3x & 0 \leq x \leq 1/2 \\ 3(1 - x) & 1/2 \leq x \leq 1. \end{cases}$$

1. Show that the tripling map preserves C, i.e. $t : C \to C$.

   As we know, the $x \in C$ iff x can be written

   $$x = \sum_{i=1}^{\infty} \frac{d_i}{3^n},$$

   where each d_i is either 0 or 2. For such an x,

   $$t(x) = 3 \sum_{i=1}^{\infty} \frac{d_i}{3^i} \mod 1 = \sum_{i=2}^{\infty} \frac{d_i}{3^{i-1}}.$$

   Note that we start the summation at $i = 2$ to ensure we get a number is $[0, 1]$. Other than that, the digits of x don't change, they simply shift. Thus, we again get a number with that type of representation.

2. Find a point of period three in C for the tripling map.

   How about $0_3\overline{002} = 1/13$?

3. Show that the tall tent map also preserves C, i.e. $T : C \to C$.

   Note that the Cantor set is contained in $[0, 1/3] \cup [2/3, 1]$. Furthermore, $T(x) = t(x)$ on $[0, 1/3]$ so we already know that

   $$T : C \cap [0, 1/3] \to C.$$

The tricky part concerns how T affects $C \cap [2/3, 1]$. On that portion, we know that $T(x) = 3(1-x)$. Let us write that function as $f \circ g$, where $f(x) = 3x$ and $g(x) = 1-x$ By the symmetry of the Cantor set,

$$g : C \cap [2/3, 1] \to C \cap [0, 1/3].$$

Then, we already know that

$$f : C \cap [0, 1/3] \to C.$$

Thus, $T : C \to C$.

4. Show T is itself a conjugacy between the two, i.e.

$$T \circ T = T \circ t,$$

when both are restricted to C.

We actually focus on the less restricted statement that T is itself a conjugacy between the two, when both are restricted to $[0, 1/3] \cup [2/3, 1]$. The common graphs of $T \circ T$ and $T \circ t$ are shown in figure 1. You can generate something like this using Desmos or Sage. Depending on how you do it, you might notice that the two graphs don't agree on the open interval $\{x \in [0, 1] : \frac{1}{3} < x\frac{2}{3}$, but that really doesn't matter since we only need the conjugacy on the Cantor set.

To prove that the conjugacy is correct, we show that both $T \circ T$ and $T \circ t$ yield the piecewise defined function suggested in the graph. For example, for $\frac{2}{3} \leq x \leq \frac{7}{9}$, it sure looks like

$$T(t(x)) = T(T(x)) = 9(x - 2/3) = 9x - 6.$$

We verify that $T(t(x))$ and $T(T(x))$ both simplify down to $9x - 6$ when $\frac{2}{3} \leq x \leq \frac{7}{9}$ using a little definition chasing. Note that

$$\frac{2}{3} \leq x \leq \frac{7}{9} \implies 0 < 3x - 2 < \frac{1}{3}.$$

Thus,

$$T(t(x)) = 3(3x - 2) = 9x - 6.$$

Similarly, $T : [\frac{2}{3}, \frac{7}{9}] \to [\frac{2}{3}, 1]$ so that

$$T(T(x)) = 3 - 3(3 - 3x) = 9 - 6x$$

on that interval as well.

The other three intervals are similar.

5. Find a point of period three in C for the tall tent map. I guess the whole point is that T maps period points of t to periodic points of t. Thus, $T(1/13) = 3/13$ should have period 3 for T. A simple check shows that

$$\frac{3}{13} \to \frac{9}{13} \to \frac{12}{13} \to \frac{3}{13}$$

under T.

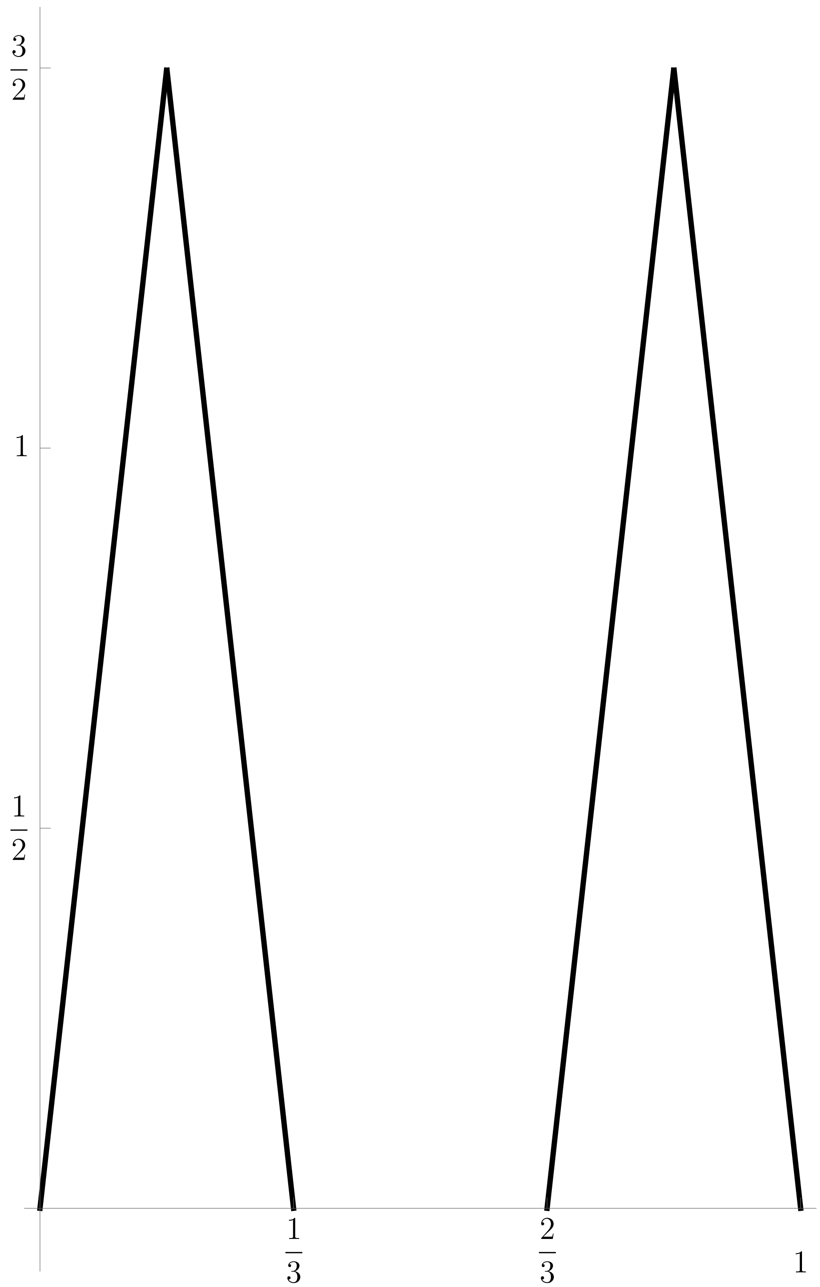

Figure 1: The common graph of $T \circ T$ and $T \circ t$