The quadratic family

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We now focus our attention on the quadratic family:

$$f_c(z) = z^2 + c.$$

First and foremost, this is a *parametrized family of polynomials*. We've already discussed both parametrized families in the context of real iteration and polynomials in the context of complex iteration. It might make sense to review some of that material:

- our class notes parametrized families of real functions
- class notes on complex polynomials, in particular:
  - The escape criterion for polynomials
  - The corollary to the escape criterion
  - The definitions of Julia set and the filled Julia set for polynomials

1 Why focus on the quadratic family?

First off, we focus on *some* parametrized family in complex iteration for the same reasons that we did so in the real case: Doing so facilitates the exploration of the range of possible behaviors that can arise in complex iteration. As it turns out, the introduction of parametrized families leads to fascinating mathematics in and of itself, like the bifurcation diagram in real iteration and its complex analog - the Mandelbrot set.

It's still fair to ask, though, "why this *particular* parametrized family"? As we've seen, the dynamics of first degree polynomials are quite simple; thus, it makes sense to move to at least quadratics. Furthermore, as it turns out, *any* quadratic is affinely conjugate to exactly one f_c. Note that by "affinely conjugate", we mean that the conjugacy function has the form $\varphi(z) = az + b$. This statement is made more precise by the following two lemmas.

Lemma 1.1. *Let $g(z) = \alpha z^2 + \beta z + \gamma$. Then $\varphi(z) = \alpha z + \beta/2$ conjugates f_c to g for*

$$c = \frac{1}{4}(2\beta - \beta^2 + 4\alpha\gamma).$$

More precisely, $\varphi \circ g = f \circ \varphi$.

Proof. Exercise 6.1!! □

Lemma 1.2. *Suppose that f_{c_1} is affinely conjugate to f_{c_2}. Then $c_1 = c_2$ and the conjugacy function is the identity function.*

Proof. Suppose that $\varphi(z) = az + b$ is an affine function such that

$$f_{c_1} \circ \varphi = \varphi \circ f_{c_2}.$$

Then

$$f_{c_1}(\varphi(z)) = (az + b)^2 + c_1 = a^2 z^2 + 2ab\, z + (b^2 + c_1)$$

and

$$\varphi(f_{c_2}(z) = a(z^2 + c_2) + b = az^2 + (ac_2 + b).$$

Equating the coefficients of z^2, we see that $a = 0$ or $a = 1$. Of course, a cannot be zero, or else φ is not a genuine affine function. Thus, $a = 1$. Equating the coefficients of z, we see that $2ab = 0$, thus $b = 0$. Equating the constant terms and taking the now known values of a and b into account, we see that $c_1 = c_2$. □

The significance of affine conjugation is illustrated in figure 1.3. On the top, we see the Julia set of $f_c(z)$ for $c = -0.468 + 0.567i$. On the bottom, we see the Julia set of

$$g(z) = \left(\frac{1}{4} - \frac{i}{4}\right) z^2 + iz - (1.57 + 1.302i).$$

The two sets are geometrically similar, because the conjugacy function

$$\varphi(z) = 2(1 + i)z + (1 - i)$$

is an affine transformation which simply rotates, scales, and shifts the plane. In fact, the Julia set on the bottom is exactly the image of the Julia set on the top after an expansion by the factor 2, a rotation through the angle 45°, and a shift by the complex number $1 - i$.

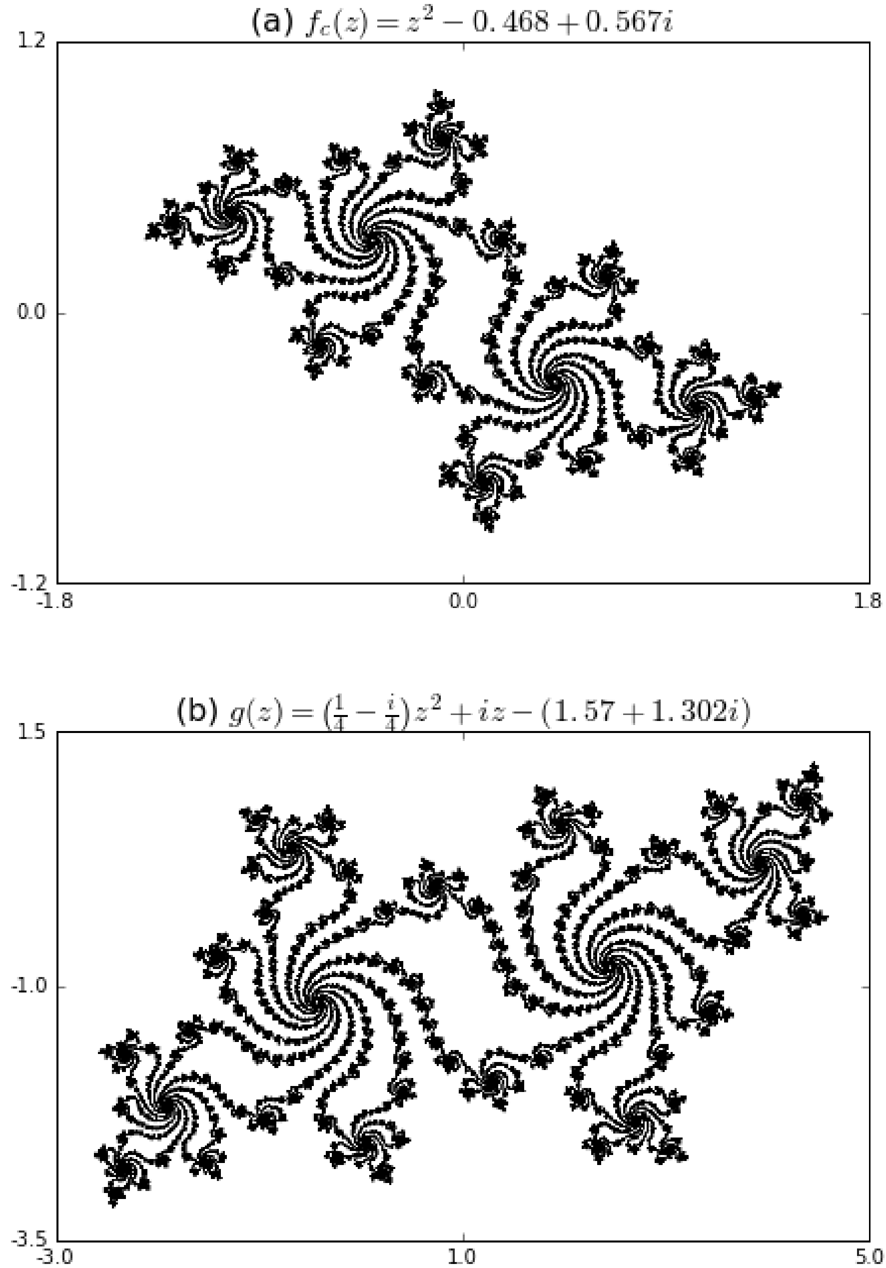

Figure 1.3: Two conjugate Julia sets

2 The quadratic escape criterion

Recall the escape radius stated in the escape criterion for general polynomials - namely

$$R = \max\left(2|a_n|, 2\frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}\right). \tag{2.1}$$

We can apply this directly to the quadratic family. It turns out, though, that we can do a bit better by focusing on this particular family.

Theorem 2.1 (The quadratic escape criterion)**.** *Let*

$$f_c(z) = z^2 + c.$$

Then, the orbit of z_0 diverges to ∞ whenever $|z_0|$ exceeds

$$R = \max(2, |c|).$$

The number R is called the escape radius *for the quadratic.*

Proof. Suppose that $z_0 \in \mathbb{C}$ satisfies $|z_0| > 2$ and $|z_0| \geq |c|$. Then, by the reverse triangle inequality,

$$\begin{aligned} |z_1| = |f(z_0)| = |z_0^2 + c| &\geq |z_0|^2 - |c| \\ &\geq |z_0|^2 - |z_0| = |z_0|\,(|z_0| - 1) = \lambda |z_0|, \end{aligned}$$

where $\lambda > 1$. Furthermore, z_1 now also satisfies $z_1 > R$ so that, by induction, $|z_n| \geq \lambda^n |z_0|$. As a result, $z_n \to \infty$ as $n \to \infty$. □

As with the polynomial escape criterion, this applies to any iterate z_i even if $z_0 \leq 2$. As a result, once some iterate exceeds 2 in absolute value, the orbit is guaranteed to escape. Note, though, that the quadratic escape criterion is a bit sharper than the version for general polynomials, when applied to the quadratic case. For example, theorem 2.1 implies that the escape radius for $f_{-2}(z) = z^2 - 2$ is 2, while equation (2.1) yields an escape radius of 4. As we already know that the Julia set of f_{-2} is exactly the interval $[-2, 2]$, this estimate is as sharp as it can be. Furthermore, under the assumption that $|c| \leq 2$, the quadratic escape radius yields a uniform radius of 2.

3 The critical orbit

Each function $f_c(z) = z^2 + c$ has exactly one critical point at the origin. It turns out that the orbit of this critical point dominates the global dynamics of the iteration of f_c.

Theorem 3.1. *Suppose we iterate f_c the critical point zero. Then, there are two mutually exclusive possibilities*

1. *the critical orbit stays bounded by the escape radius, in which case the Julia set and filled Julia sets are connected, or*

2. *the critical orbit diverges to ∞, in which case the Julia set is a Cantor set.*

For the time being, we will content ourselves with a proof that the Julia set is totally disconnected when $|c| > 2$ and sketch of the rest. The general idea is based on a process called *inverse iteration.* Given and function $f : \mathbb{C} \to \mathbb{C}$ and a set $S \in \mathbb{C}$ let

$$f^{-1}(S) = \{z \in \mathbb{C} : f(z) \in S\}.$$

The set $f^{-1}(S)$ is called the *inverse image* or *pre-image* of S. We define an initial approximation J_0 and let $J_n = f_c^{-n}(J_0) \equiv f_c^{-1}(J_{n-1})$. As we will see, if the initial approximation is chosen correctly, the sequence of iterates will collapse down to filled Julia set.

Proof. Suppose that $|c| > 2$ and consider the disk D of radius $|c|$ centered at the origin. By the proof of theorem 2.1, every point on the boundary of this disk maps to a point whose absolute value is larger than $|c|$. As a result, the image of the boundary is a loop that completely encircles D. (In fact, it's a circle of radius $|c|^2$ centered at c, as you'll show in exercise 6.2.) Thus, $f(D) \supset D$. As a result, $f^{-1}(D) \subset D$. We now perform inverse iteration from D.

To help us understand $f^{-1}(D)$, note that $f_c(0) = c$, which lies on the boundary of D. In fact, $f_c(z) = z^2 + c = c$ is equivalent to $z^2 = 0$ so that zero is the *only* point that maps to c. Every other point on the boundary of D has exactly two pre-images. Thus, the pre-image of the boundary of D is a figure 8 with a cut point at the origin. The pre-image of D itself consists of two lobes bounded by the two curves forming the figure 8.

We next consider $f_c^{-2}(D)$. This set consists of two separate pieces, one in each of the lobes of $f_c^{-1}(D)$. More generally, $f_c^{-n}(D)$ consists of 2^{n-1} obtained by cutting the pieces of $f_c^{-(n-1)}$ in half.

Finally, the filled Julia set itself is exactly

$$\bigcap_{n=1}^{\infty} f_c^{-n}(D).$$

As the diameters of these sets decrease down to zero, the filled Julia set coincides exactly with the Julia set. The construction above yields an obvious correspondence between the points in the Julia set and the points in the Cantor set.

The construction just described in illustrated in figure 3.2(a). □

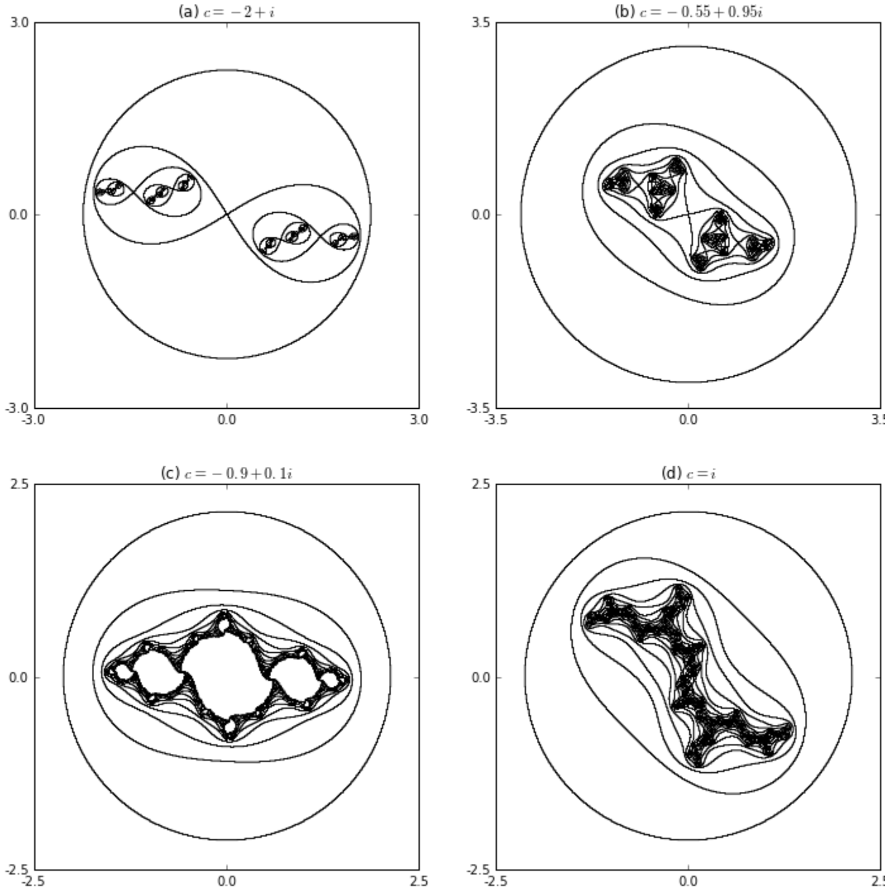

Figure 3.2: Inverse images collapsing to filled Julia sets

The situation just described is more complicated when $|c| \leq 2$. If $|c| \leq 2$ and the critical orbit escapes, then there is some first value of n with the property that $|f_c^n(0)| > 2$. We can then apply the inverse iteration idea to the disk of radius $|f_c^n(0)|$ centered at the origin. The figure 8 configuration then appears after n iterates and the ideas of the proof are then applicable. This situation is illustrated in figure 3.2(b).

If the critical orbit does not escape, then we can attempt inverse iteration with a disk D whose radius is larger than two. In this case, the figure 8 never forms so that $f^{-n}(D)$ is a connected set for all n. As a result, the filled Julia set is a connected set since it's the intersection of a nested family of connected sets. This set is illustrated in 3.2(c) and (d). In figure (c), f_c has an attractive orbit. In figure (d), f_c does not have an attractive orbit so that the sets $f^{-n}(D)$ collapse down to have area zero. Since any attractive behavior must attract the critical orbit, one easy way to ensure the lack of attractive behavior is to chose c so that the orbit of zero lands exactly on a repelling orbit. In fact, one can prove that this always happens when zero is pre-periodic. In 3.2(d), $c = i$ and zero eventually maps to an orbit of period 2.

4 The Mandelbrot set

Theorem 3.1 expresses a fundamental dichotomy amongst quadratic Julia sets. There are two types: connected and totally disconnected. Furthermore, the type that arises from a particular value of c depends completely on the orbit of the critical point. Thus, it might be fruitful to partition the parameter plane into two portions: the portion where the critical orbit stays bounded and the portion where the critical orbit diverges to infinity. This partition leads to one of the most iconic images from the study of chaos - the Mandelbrot set.

Definition 4.1 (The Mandelbrot set)**.** The *Mandelbrot set* is the set of all complex parameters c such that the Julia set of the function $f_c(z) = z^2 + c$ is connected. Equivalently, by theorem 3.1, it is the set of all c values such that the orbit of zero under iteration of f_c is bounded by 2

Theorem 3.1 yields an algorithm to generate images of the Mandelbrot set that's very similar to our algorithm for generating Julia sets of polynomials.

Algorithm 4.2 (The escape time algorithm the Mandelbrot set)**.**

1. *Choose some rectangular region in the complex plane bound on the lower left by, say,* c_{min} *and on the upper right by* c_{max}*.*
2. *Partition this region into large number of rows and columns. We'll call the intersection of a row and column a "pixel", which corresponds to a complex value of the parameter* c*.*
3. *For each pixel, iterate the corresponding function* f_c *from the origin until one of two things happens:*
   (a) *We exceed the escape radius* 2 *in absolute value, in which case we shade the pixel according to how many iterates it took to escape.*
   (b) *We exceed some pre-specified maximum number of iterations, in which case we color the pixel black.*

The results of this algorithm are shown in figure 4.3. While the algorithm is very similar to our algorithm for generating Julia sets, it's very important to understand the distinctions. The Mandelbrot set lives in the *parameter plane*, while Julia sets live in the *dynamical plane*. For the Julia set of a quadratic function f_c, we fix the value of c and explore the behavior of the orbit of z_0 for a grid of choices of z_0. For the Mandelbrot set, we explore the global behavior of f_c by examining the critical orbit; we do this for a grid of choices of c.

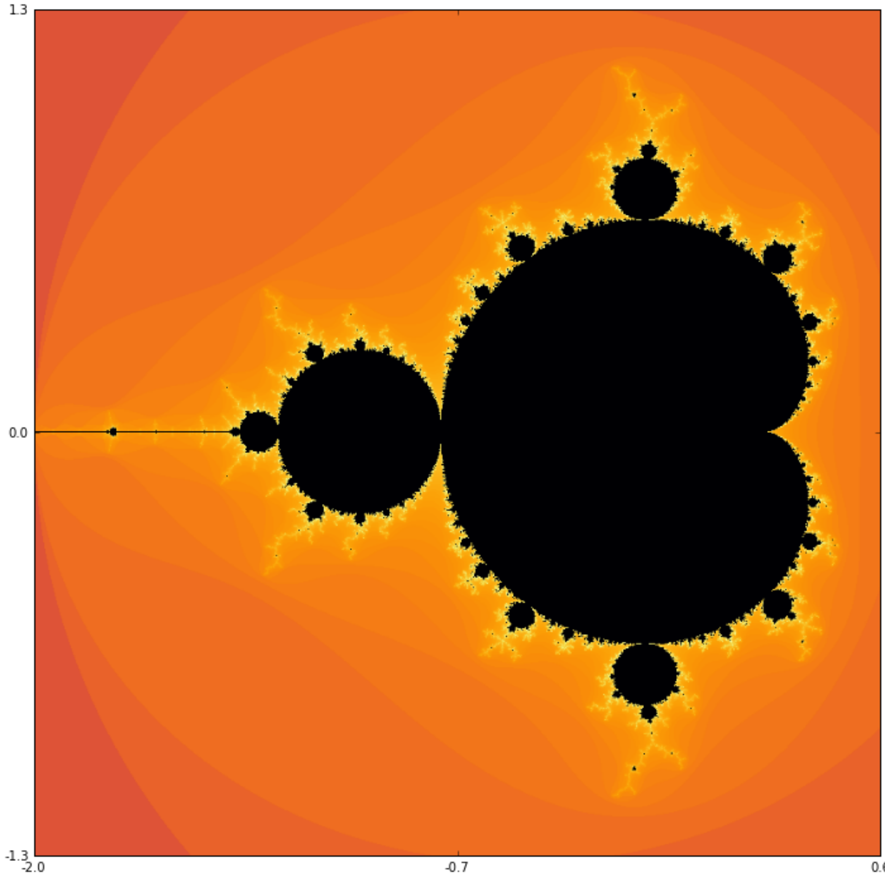

Figure 4.3: The Mandelbrot set

5 The components of the Mandelbrot set

By definition, the Mandelbrot set decomposes the complex parameter plane into two components corresponding to a very coarse decomposition of the types of behavior of f_c. A glimpse at figure 4.3 (or any of the many images of it available on the web) shows that it consists of rather conspicuous subparts. It appears that there is a large "main cardioid" and that, attached to that main cardioid are a slew of disks or near disks. If we zoom in, we see many smaller copies of the whole set. As it turns out, all these components correspond to particular dynamical behavior. Values of c chosen from within one component yields functions f_c with similar qualitative behavior. As c moves from one component to another, that qualitative behavior changes; we say that a *bifurcation* occurs. In this section, we'll describe the most conspicuous behavior that we see.

5.1 The main cardioid

The interior of the main cardioid consists of all those values of c with the property that f_c has an attractive fixed point. This can be characterized algebraically as

$$f_c(z) = z \tag{5.1}$$
$$|f_c'(z)| < 1. \tag{5.2}$$

Now, on the *boundary* we should have

$$f_c(z) = z \tag{5.3}$$
$$|f_c'(z)| = 1. \tag{5.4}$$

That is, the fixed point is no longer strictly attractive on the boundary but just neutral. Since any complex number of absolute value 1 can be expressed in the form $e^{2\pi it}$, this pair of equations can be written without the absolute value:

$$f_c(z) = z \tag{5.5}$$
$$f_c'(z) = e^{2\pi it}, \tag{5.6}$$

for some t. Taking into account the fact that $f_c(z) = z^2 + c$, we get

$$z^2 + c = z \tag{5.7}$$
$$2z = e^{2\pi it}. \tag{5.8}$$

This system of equations can be solved for z and c in terms of t. It's not even particularly hard. The second equation yields $z = e^{2\pi it}/2$. This can be plugged into the first equation to get

$$c = e^{it}/2 - e^{2it}/4.$$

This happens to be the parametric representation of a cardioid and is exactly the formula used to generate figure 5.1

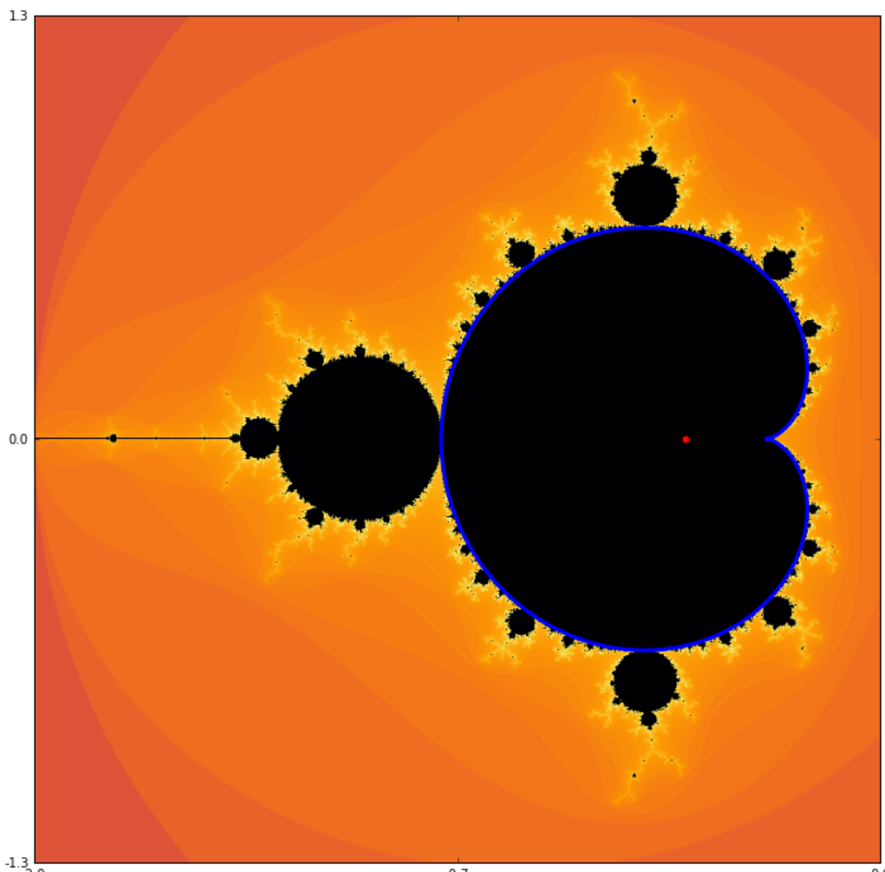

Figure 5.1: The Mandelbrot set with the main cardioid highlighted

5.2 The period two bulb

After the main cardioid, the next largest component is the disk attached to the main cardioid at the point $c = -3/4$. This is, in fact, an actual disk attached at that exact point. We can prove this in much the same way that we derived the formula for the main cardioid, though the algebra is a little more involved.

This disk is known as the *period two bulb*; every function f_c where c is chosen from the period two bulb has an attractive orbit of period two. We can characterize this algebraically by first defining $F_c(z) = f_c(f_c(z))$. A point is then part of an attractive orbit of period 2 for f_c if

$$F_c(z) = z \tag{5.9}$$
$$|F_c'(z)| < 1. \tag{5.10}$$

Mimicking the construction of the main cardioid in the period 1 case, we find that z and c must satisfy

$$F_c(z) = z \tag{5.11}$$
$$F_c'(z) = e^{2\pi it}, \tag{5.12}$$

for some t. Taking into account the fact that $F_c(z) = (z^2 + c)^2 + c$, we get

$$(z^2 + c)^2 + c = z \tag{5.13}$$
$$4z(z^2 + c) = e^{2\pi it}. \tag{5.14}$$

While clearly this is a bit more involved than the period one case, a computer algebra system makes quick work of it. Listing 5.2 shows how to solve this system with Mathematica and listing 5.3 shows how to solve this system with the Python library SymPy.

```
f[c_][z_] = z^2 + c;
F[c_][z_] = Nest[f[c], z, 2];
eqs = {F[c][z] == z, F[c]'[z] == Exp[2 Pi*I*t]};
Expand[c /. Last[Solve[eqs, {c, z}]]]
(* Out:
        -1 + E^((2*I)*Pi*t)/4
*)
```

Listing 5.2: Mathematica code to parametrize the period two bulb

```
from sympy import *
c,z,t = var('c,z,t')
def f(c,z): return z**2+c
def F(c,z): return f(c,f(c,z))
eqs = [F(c,z)-z,diff(F(c,z),z)-exp(2*pi*I*t)]
sol = solve(eqs,(c,z), dict=True)[0]
print(expand(sol[c]))
```

```
# Out:
exp(2*I*pi*t)/4 - 1
```

Listing 5.3: Python code to parametrize the period two bulb

Note that both program listings 5.2 and 5.3 yield the result

$$c = -1 + \frac{1}{4}e^{2\pi i t},$$

which parametrizes a circle of radius 1/4 centered at the point −1.

Also, it's certainly feasible to solve this system by hand. The expression $F_c(z) - z$ factors so that the system really just involves a couple of quadratics. Use of the computer will be quite convenient as we move into higher order examples, though.

5.3 The Mandelbrot set as a bifurcation locus

We've now got a solid understanding of the period 1 component of the Mandelbrot set (the main cardioid) and the period 2 component (the period 2 bulb). The period 2 component is called a "bulb" because it is directly attached to the main cardioid. Indeed, using our parametrizations, you can show easily enough that the point $c = -3/4$ lies on both the boundary of the main cardioid and on the boundary of the period 2 component. Note that, for this value of c, f_c has a neutral fixed point at $z = -1/2$. As c moves from the main cardioid, through the point $-3/4$, and into the period 2 bulb, the function f_c undergoes a bifurcation - that is a qualitative change in the global dynamics of the function. More specifically, f_c changes from having an attractive fixed point to having an attractive orbit of period 2. This is very much like the bifurcation diagram that we saw when studying real iteration. In fact:

> The Mandelbrot set is a version of the bifurcation diagram in the context of complex dynamics, rather than real dynamics.

The geometric relationship between the Mandelbrot set and the bifurcation diagram is made explicit in figure 5.4.

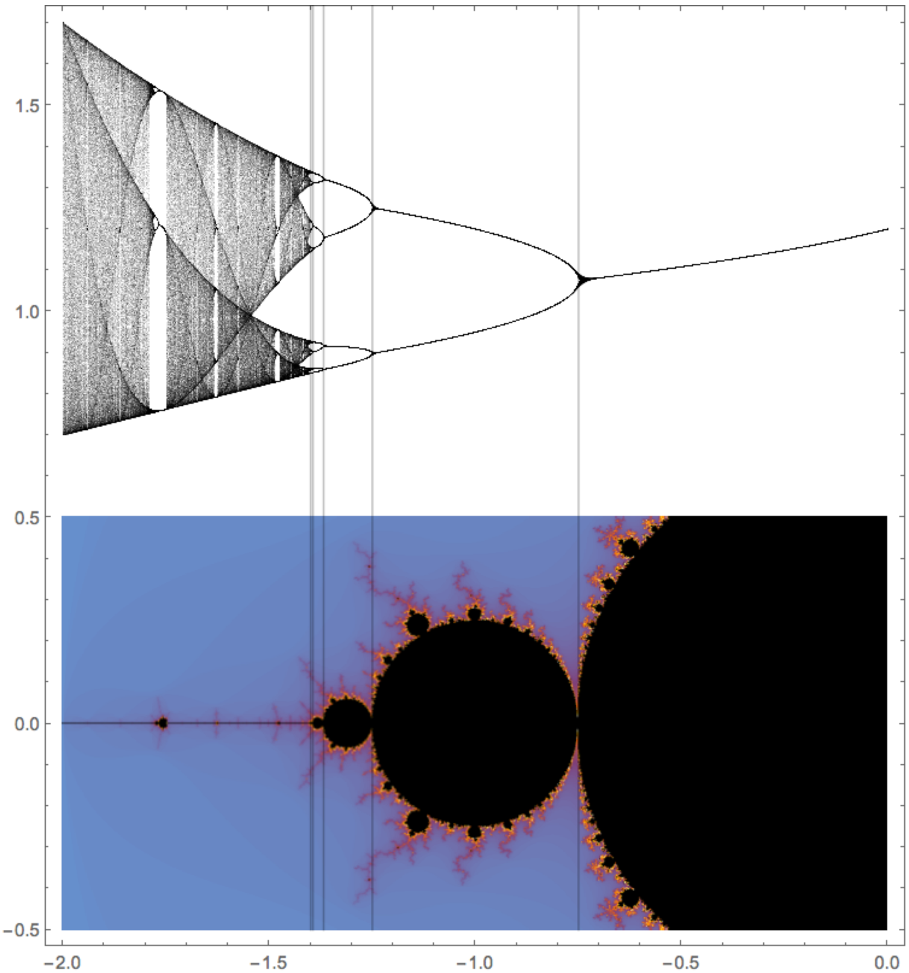

Figure 5.4: Two Mandelbrot set aligned with the bifurcation diagram

Figure 5.4 shows only how the Mandelbrot set aligns with the real bifurcation diagram. There is really much more to the picture, though. All the visible components of the Mandelbrot set correspond to distinct, qualitative behavior. As c moves within one component, the qualitative behavior of f_c stays the same. More precisely, any two values of c chosen from the same component will have attractive orbits of the same period. (There should, however, be some degree of *quantitative* difference, as we know from lemma 1.2.) As c moves from any component of the Mandelbrot set to another in the complex plane, f_c undergoes a bifurcation - period doubling, tripling, or whatever. If c exits the Mandelbrot set, f_c undergoes a more complicated type of bifurcation.

This all suggests another way to draw the Mandelbrot set. We iterate from the critical point zero, as in algorithm 4.2, but we stop iteration whenever we find some sort of attractive behavior. We then color the point c according to the period of the orbit we find and shade it according to how long it took to detect the periodicity. The result is shown in figure 5.5.

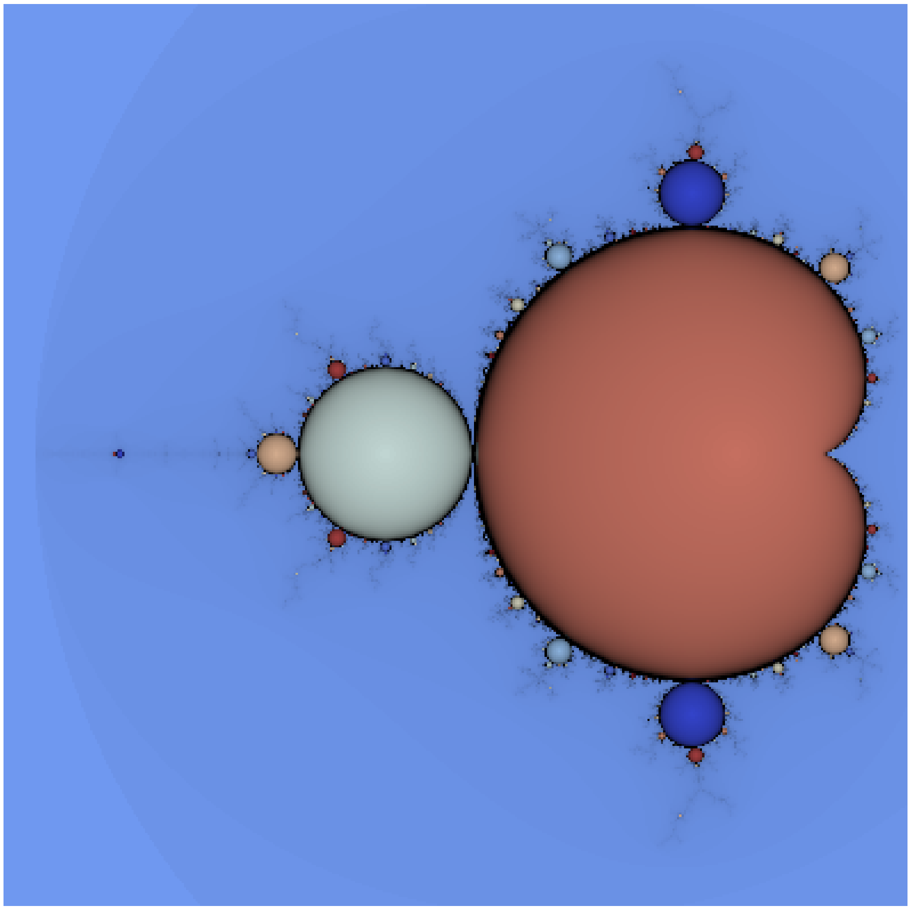

Figure 5.5: The components of the Mandelbrot set shaded according to periodicity

5.4 Higher period bulbs

Figure 5.5 shows that there are lots of disk-like bulbs attached to the main cardioid. As c moves from the main cardioid into one of these bulbs, f_c bifurcates from having an attractive fixed point, to an orbit of period q for some integer $q > 1$. After the period 2 bulb, the two next largest bulbs near the top and bottom of the main cardioid are period 3 bulbs. The other bulbs have larger period still.

Let's try to gain a quantitative understand of the nature of these bulbs. To do so, we re-consider our description of the main cardioid from subsection 5.1. Again, the interior of the main cardioid can be characterized algebraically as

$$f_c(z) = z \tag{5.15}$$

$$|f_c'(z)| < 1. \tag{5.16}$$

More generally, though, we can obtain a fixed point with any given multiplier γ that we like by solving

$$f_c(z) = z \tag{5.17}$$

$$f_c'(z) = \gamma. \tag{5.18}$$

If we solve this system for c and z, we obtain

$$c = \frac{1}{4}\left(2\gamma - \gamma^2\right) \tag{5.19}$$

$$z = \gamma/2. \tag{5.20}$$

Thus, f_c has a fixed point with multiplier γ at $\gamma/2$ when $c = (2\gamma - \gamma^2)/4$.

Let us now define $c(\gamma) = (2\gamma - \gamma^2)/4$ and consider the image of rays of the form $r \exp(2\pi i\theta)$, where θ is fixed and $0 \leq r \leq 1$. These should all lie within the main cardioid and stretch from zero to a point on the boundary of the cardioid. Some of these are plotted in figure 5.6.

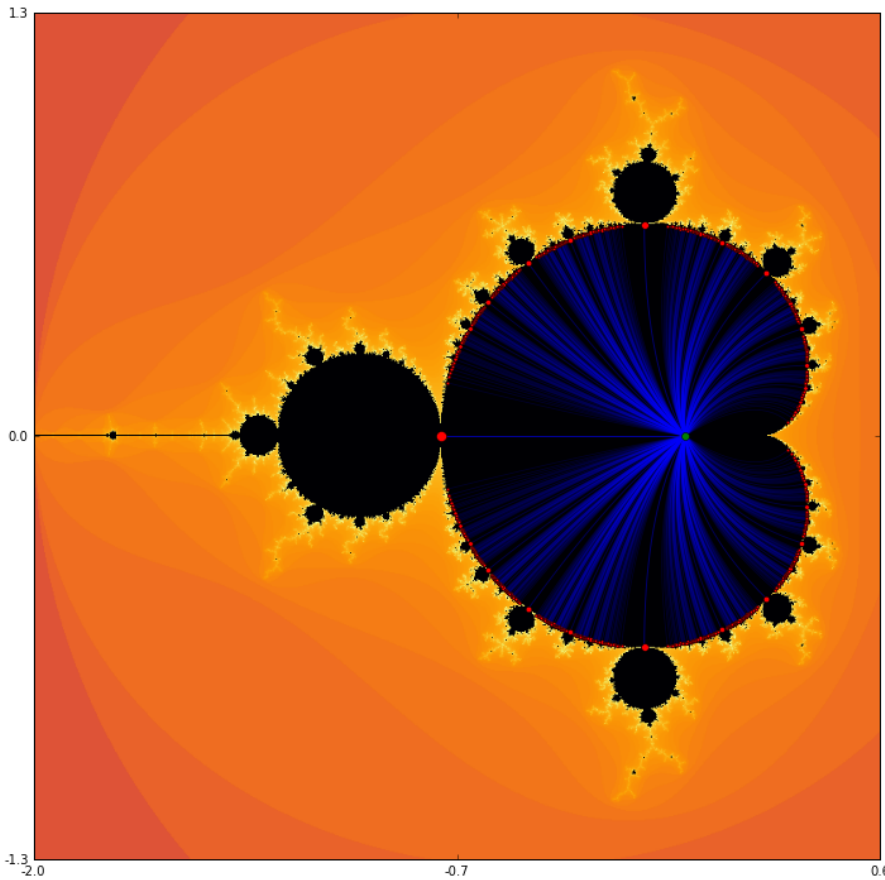

Figure 5.6: Bulbs located by the cardioid parametrization

Note that the rays shown in figure 5.6 all correspond to rational values of θ and, as it appears, a periodic bulb is attached to the main cardioid at every rational ray. Even more is true - if $\theta = p/q$, then the bulb attached to the main cardioid at the point $c(\exp(2\pi i\theta))$ has period q and rotation number p.

To understand this, recall that $c(\gamma)$ is built specifically so that $f_{c(\gamma)}$ will have multiplier γ. So, what happens to the attractive orbits of $f_c(\gamma)$ for

$$\gamma = re^{2\pi i\theta}$$

for a fixed rational value of θ as r moves from zero up to one and even just a little bit past? For $r < 1$, f_c has an attractive fixed point. For $r = 1$, f_c has a neutral fixed point with multiplier $\gamma = \exp(2\pi i\theta)$. Note that the geometric effect of multiplying by this particular value of γ is to rotate through the angle $2\pi\theta = 2\pi p/q$. If we do this q times, we've rotated through the angle $2\pi\theta = 2\pi p$, which returns to where we started. We might expect an attractive orbit of period q; we don't have it yet because the origin still in the basin of the neutral fixed point. Once $r > 1$, that fixed point becomes slightly repulsive and the attractive orbit of period q appears.

Note that each of these bulbs has further bulbs sprouting off and similar considerations apply.

5.5 Baby-brots

There are loads of little copies of the Mandelbrot set strewn throughout. For the time being, I'll refer to this Math.StackExchange post: http://math.stackexchange.com/questions/404066/.

6 Exercises

1. Prove lemma 1.1

2. Show that the image of the circle of radius $|c|$ centered at c under f_c is the circle of radius $|c|^2$ centered at the origin.

In the next couple of problems, we'll try to get a grip on the family of functions $g_\lambda(z) = \lambda z + z^2$.

3. The escape radius

(a) Use the triangle inequality to show that $|g_\lambda(z_0) \geq |z_0|$, whenever $|z_0| > 2\lambda$. Conclude that the orbit of z_0 escapes whenever an iterate excedes 2λ.

(b) How does this compare to our general polynomial escape criterion?

4. The *escape locus* for g_λ (or any family of quadratics) is a partition of the complex parameter plane into two regions - one where the critical orbit stays bounded and one where the critical orbit diverges. This escape locus is shown in figure 6.1. Let's try to understand a couple of things about this image.

(a) Show that the origin is an attractive (or super-attracitve) fixed point of g_λ, whenever $|\lambda| < 1$. This observation yields what prominent feature in figure 6.1?

(b) Let $\varphi(z) = z + (1 - \lambda)$. Show that

$$g_\lambda \circ \varphi = \varphi \circ g_{2-\lambda}.$$

How is this observation related to the symmetry that we see in figure 6.1?

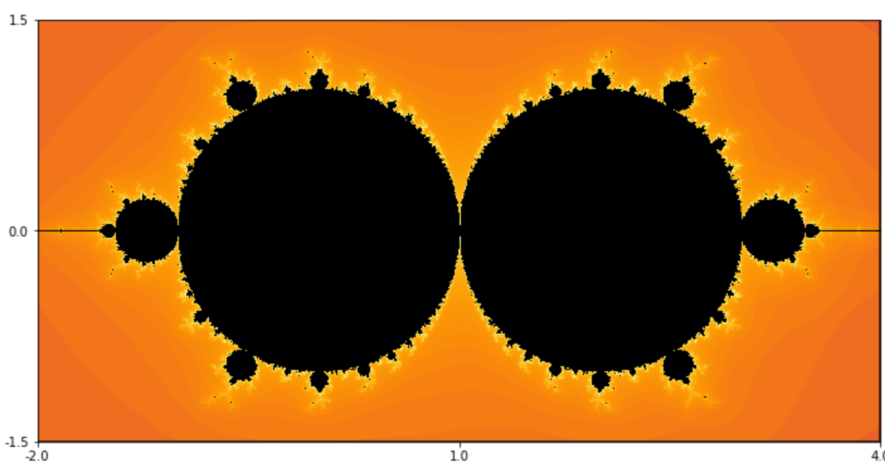

Figure 6.1: The escape locus for $g_\lambda(z) = \lambda z + z^2$.