Notes on real iteration

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Introduction

Here are some introductory notes on the study of discrete, real dynamics. That is, we iterate a function $f : \mathbb{R} \to \mathbb{R}$. The objective is to introduce just the basics of discrete dynamics at the undergraduate level with a focus on topics that will be useful in the study of complex dynamics.

1 Basic notions

We begin with some of the most fundamental definitions and examples. While these definitions are stated for real functions, many of them extend quite easily to other contexts.

Definition 1.1. Let $x_0 \in \mathbb{R}$ be an initial point and define a sequence (x_n) recursively by $x_{n+1} = f(x_n)$. This sequence is called *the orbit* of x_0 under iteration of f.

Some orbits don't move; they are fixed.

Definition 1.2. A point $x_0 \in \mathbb{R}$ is a *fixed point* of f if $f(x_0) = x_0$.

Sometimes an orbit might return to the original starting point.

Definition 1.3. Suppose that the orbit (x_n) satisfies

$$x_0 \to x_1 \to x_2 \cdots \to x_{n-1} \to x_0$$

and $x_n = x_0$. Such an orbit is called a *periodic orbit* and the points themselves are called *periodic points.* If $x_k \neq x_0$ for $k = 1, 2, \ldots, n-1$, then n is called the *period* of the orbit.

Note that a fixed point is a periodic point with period one.

Sometimes, the orbit of a non-periodic point might land on a periodic orbit.

Definition 1.4. If the zeroth term x_0 of an orbit (x_n) is not periodic but x_n is periodic for some n, then x_0 and its orbit are called *pre-periodic.*

Example 1.5. Let $f(x) = x^2 - 1$. Then zero is a periodic point and one is a pre-periodic point, as the reader may easily verify.

To find a fixed point, we can simply set $f(x) = x$ and solve the resulting equation. In this case, we get

$$x^2 - 1 = x \text{ or } x^2 - x - 1 = 0.$$

We can then apply the quadratic formula to find that

$$x = \frac{1 \pm \sqrt{5}}{2}$$

are both fixed.

Often, it helps to express these ideas in terms of composition of functions. We denote the n fold composition of a function with itself by f^n. That is, $f^2 = f \circ f$ and $f^n = f \circ f^{n-1}$. (Be careful note to confuse this with raising a function to a power.) A more complete understanding of periodicity arises from the study of the functions f^n. For example, a point x_0 has period n iff $f^n(x_0) = x_0$ but $f^k(x_0) \neq x_0$ for $k = 1, 2, \ldots, n-1$.

2 Graphical analysis

There is an efficient geometric tool to visualize functional iteration. The basic idea is simple: Suppose we graph the function f together with the line $y = x$. If those two graphs intersect; that point of intersection is a fixed point. Now, suppose we're on the line at the point (x_i, x_i). If we move vertically to the graph of the function, we preserve the x coordinate but change the y coordinate to $f(x_i)$. Thus, we arrive at the point $(x_i, f(x_i)) = (x_i, x_{i+1})$. If we then move horizontally back to the line $y = x$ we now preserve the y coordinate but change the x coordinate so that the x and y coordinates are the same. Thus, we arrive at the point (x_{i+1}, x_{i+1}).

In summary: The process of moving vertically from a point on the line $y = x$ to the graph of f and back to the line horizontally is a geometric representation of one application of the function f. This step is illustrated in figure 2.1(a). Repeated application of this process represents repeated application of f, i.e. iteration. This is illustrated in figure 2.1(b). Note that the orbit appears to be attractive.

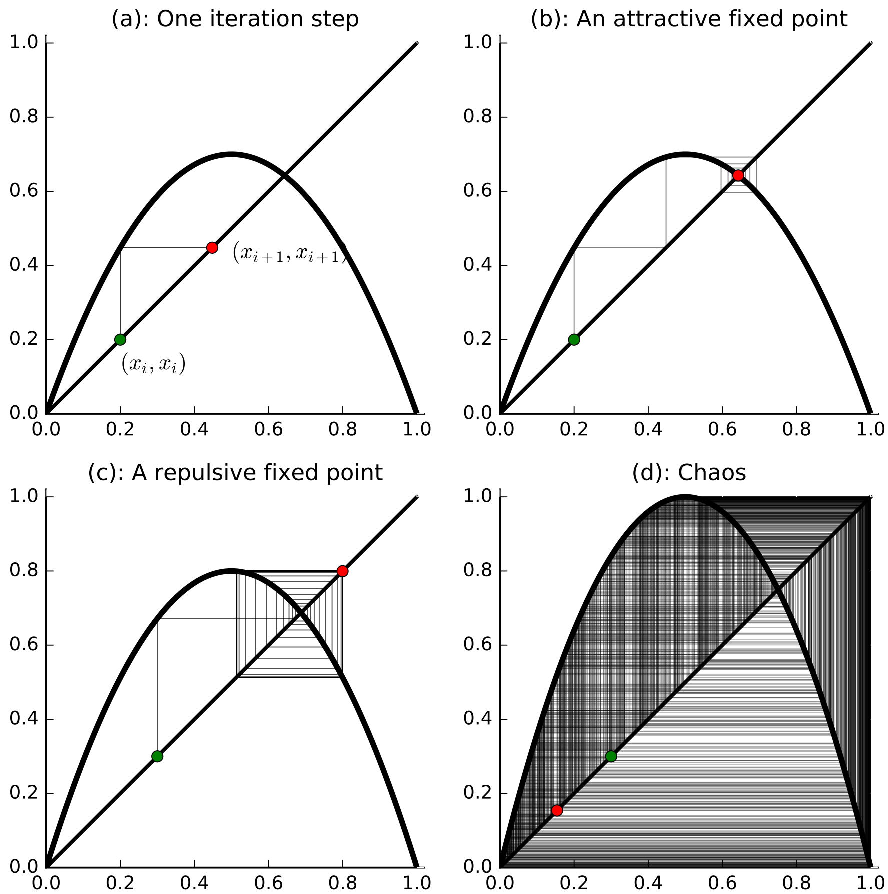

Figure 2.1: Some cobweb plots

It turns out that the process is quite sensitive to the slope of the function at the point of intersection. A slightly steeper function is shown in figure 2.1(c); we notice that the fixed point now appears to be repelling. Finally, figure 2.1(d) illustrates the fact that all hell can break loose.

3 The classification of fixed points

The cobweb plots in the previous section illustrate that the slope of the function at the point where it crosses the fixed point might play a role in the behavior of the iterates near that fixed point. We explore that further here. First, we explore the simplest situation - functions with constant slope.

Example 3.1 (Linear iteration)**.** Suppose that f is a linear function: $f(x) = ax$. It's easy to see that the origin $x = 0$ is a fixed point of f. Show that any non-zero initial point x_0 moves away from the origin under the iteration of f whenever $|a| > 1$ but moves towards the origin under iteration of f if $|a| < 1$.

Solution. This is easy, once we recognize that there is a closed form for the n^{th} iterate of f, namely $f^n(x) = a^n x$. Note that cobweb plots for these functions are shown in figure 3.2.

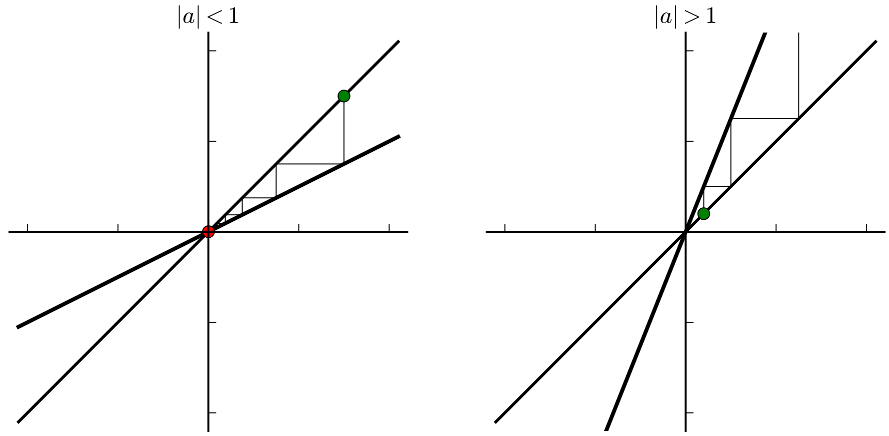

Figure 3.2: Some linear cobweb plots

Exercise 3.3 (Affine iteration)**.** Suppose, that f is an *affine function*, which just means that it has the form $f(x) = ax + b$, where $a \neq 0$. Suppose also that $x_0 \in \mathbb{R}$ and let's consider the iterates $x_{n+1} = f(x_n)$

1. Show that f has a unique fixed point iff $a \neq 1$. What if $a = 1$?
2. Suppose that $|a| < 1$. Show that the sequence of iterates converges to the fixed point of f.
3. Suppose that $|a| > 1$. Show that the sequence of iterates diverges.
4. What happens if $a = -1$?

Example 3.1 and exercise 3.3 together classify the dynamical behavior of first order polynomials completely and show that their behavior is fairly simple. For that reason, we focus on polynomials of degree two and higher. Already in the quadratic case, we can find much more complicated and interesting behavior. Motivated by the behavior we see in linear and affine functions, we make the following defintion.

Definition 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $x_0 \in \mathbb{R}$ is a fixed point of f. Then we classify x_0 as

1. *attractive*, if $0 < |f'(x_0)| < 1$,
2. *super-attractive*, if $f'(x_0) = 0$,
3. *repulsive* or *repelling*, if $|f'(x_0)| > 1$, or
4. *neutral*, if $|f'(x_0)| = 1$,

The number, $f'(x_0)$ is called the *multiplier* for the fixed point. If, in the attractive case, the multiplier is zero, we say that x_0 is *super-attractive*.

The following theorem justifies this notation.

Theorem 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $x_0 \in \mathbb{R}$ is a fixed point of f.*

1. *If x_0 is an attractive or super-attractive fixed point for f, then there is an $\varepsilon > 0$ such that the orbit of x under iteration of f tends to x_0 for every x such that $|x - x_0| < \varepsilon$.*

2. *If x_0 is a repelling fixed point for f, then there is an $\varepsilon > 0$ such that the orbit of x under iteration of f tends (initially) away from x_0 for every x such that $|x - x_0| < \varepsilon$.*

Proof. We prove part one; the second part is similar. Since $|f'(x_0)| < 1$ and f' is continuous, we may choose an $\varepsilon > 0$ and a postive number $r < 1$ such that $|f'(x)| < r$ for all x such that $|x - x_0| < \varepsilon$. Then, given x such that $|x - x_0| < \varepsilon$, we can apply the Mean Value Theorem to obtain a number c such that

$$|f(x) - x_0| = |f(x) - f(x_0)| = |f'(c)||x - x_0| \leq r\varepsilon.$$

By induction, we can show that

$$|f^n(x) - x_0| \leq r^n \varepsilon.$$

The result follows, since $r^n \varepsilon \to 0$ as $n \to \infty$. $\square$

From the proof, we see that $x_n \to x_0$ exponentially and that the magnitude of $|f'(x_0)|$ dictates the base of that exponential. When $f'(x_0) = 0$, the rate is faster than exponential.

Example 3.6. The function $f(x) = 4.8\,x^2(1 - x)$ is graphed in figure 3.7, along with the line $y = x$. The points of intersection are fixed points and, from left to right, they are super-attractive, repulsive, and attractive. The reader should consider the appearance of a cobweb plot for initial values starting near each of those fixed points.

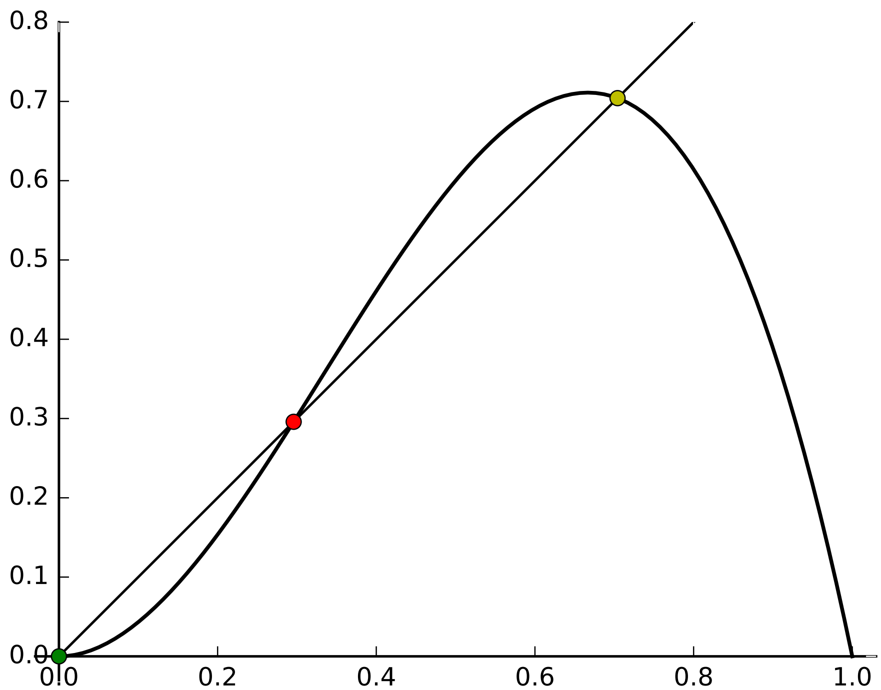

Figure 3.7: Three types of fixed points

The behavior of iterates near a neutral fixed point can be more varied.

Exercise 3.8 (Dynamical behavior near neutral fixed points)**.** For each of the following scenarios, find an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a fixed point x_0 of f satisfying that scenario.

1. There is an $\varepsilon > 0$ such that the orbit of x tends to x_0 for all x such that $|x - x_0| < \varepsilon$.
2. There is an $\varepsilon > 0$ such that the orbit of x tends initially away from x_0 for all x such that $|x - x_0| < \varepsilon$.
3. There is an $\varepsilon > 0$ such that the orbit of x tends to x_0 for all x such that $0 < x - x_0 < \varepsilon$ but the orbit of x tends initially away from x_0 for all x such that $0 < x_0 - x < \varepsilon$.
4. There is an $\varepsilon > 0$ such that the orbit of x tends to x_0 for all x such that $0 < x - x_0 < \varepsilon$ but the orbit of x tends initially away from x_0 for all x such that $0 < x_0 - x < \varepsilon$.

4 Classification of periodic orbits

As mentioned at the end of section one a periodic point for f of period n is a fixed point of f^n. Treating the points of a periodic orbit this way allows us to extend the classification as fixed points to periodic orbits.

Definition 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $x_0 \in \mathbb{R}$ is a periodic point of f with period n. Let $F = f^n$. We classify x_0 and its orbit as

1. *attractive*, if $|F'(x_0)| < 1$,
2. *super-attractive*, if $F'(x_0) = 0$,
3. *repulsive* or *repelling*, if $|F'(x_0)| > 1$, or
4. *neutral*, if $|F'(x_0)| = 1$,

The number $F'(x_0)$ is called the *multiplier* of the orbit. If, in the attractive case, the multiplier is zero, we say that the orbit is *super-attractive*.

There is a nice characterization of the multiplier of an orbit that allows us to compute it without explicitly computing a formula for f^n.

Lemma 4.2. *Suppose that*

$$x_0 \to x_1 \to x_2 \to \cdots \to x_{n-1} \to x_0$$

is an orbit of period n for $f : \mathbb{R} \to \mathbb{R}$. Then the multiplier of the orbit is

$$f'(x_0)f'(x_1)\cdots f'(x_{n-1}).$$

Proof. First note that for an $n = 2$, we can apply the chain rule to obtain

$$\frac{d}{dx}f^2(x) = \frac{d}{dx}f(f(x)) = f'(f(x))f'(x).$$

Thus, if $x_0 \to x_1 \to x_0$ is an orbit of period two and we evaluate that equation at x_0, we obtain

$$\left.\frac{d}{dx}f^2(x)\right|_{x=x_0} = f'(x_1)f'(x_0).$$

The result for orbits longer than two can be proven by induction, since

$$\frac{d}{dx}f^n(x) = \frac{d}{dx}f(f^{n-1}(x)) = f'(f^{n-1}(x))\frac{d}{dx}f^{n-1}(x).$$ □

Note that the only way the product in 4.2 is zero, is if one of the terms is zero. This yields the following corollary.

Corollary 4.3. *A periodic orbit is super-attracting if and only if it contains a critical point.*

Example 4.4. Let $f(x) = x^2 - 1$. Note that $f(0) = -1$ and $f(-1) = 0$ so that $0 \to 1 \to 0$ forms an orbit of period 2. To see if this orbit is attractive, we examine

$$F(x) = f(f(x)) = (x^2 - 1)^2 - 1 = x^4 - x^2.$$

Note that $F'(0) = 0$ and $F'(-1) = 0$; thus, the orbit is super-attractive.

The plots of f and f^2, together with $y = x$, are shown in figure 4.5. Note that f has two fixed points shown in red. They can found by solving the equation $x^2 - 1 = x$ and they are both repulsive under iteration of f. The two super-attractive orbits of f^2 are shown in green.

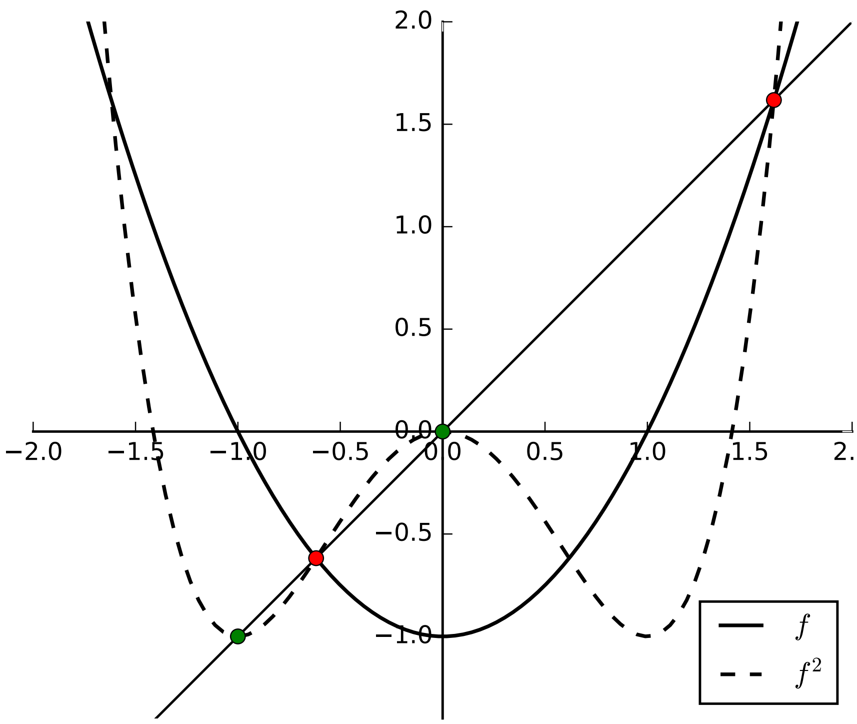

Figure 4.5: An attractive orbit of period two

5 Parametrized families of functions

Rather than explore the behavior of a single function at a time, we can introduce a parameter and explore the range of behavior that arises in a whole family of functions. Two important examples are

1. *The quadratic family*: $f_c(x) = x^2 + c$
2. *The logistic family*: $f_\lambda(x) = \lambda x(1 - x)$

The cobweb plots shown back in figure 2.1 are all chosen from the logistic family with $\lambda = 2.8$, $\lambda = 3.2$, and $\lambda = 4$. Even in those three pictures with graphs that look so very similar, we see three different types of behavior: an attractive fixed point, an attractive orbit of period two, and chaos (which can be given a very technical meaning.

Figure 5.1 shows some cobweb plots for the quadratic family of functions. Note that the behavior we see is very similar to the behavior we see for the logistic family - a fact that will become more understandable once we study conjugacy in section 6

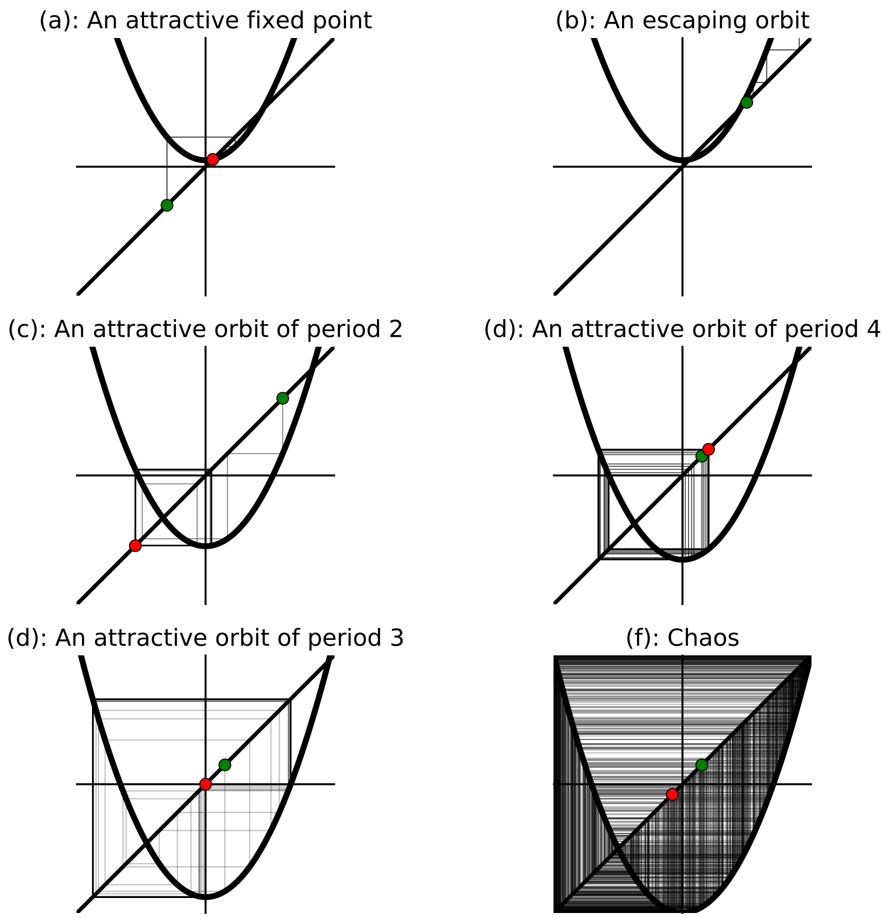

Figure 5.1: Some cobweb plots for the quadratic family

5.1 The bifurcation diagram

A fabulous illustration of the types of behavior that can arise in a family of functions indexed by a single parameter and each with a single critical point can be generated as follows: For each value of the parameter, compute a large number points of the orbit of the critical point (maybe 1000 iterates). Since we're interested in long term behavior, rather than any transient behavior, discard the first few iterates (maybe 100). Then, plot the remaining points in a vertical column at the horizontal position indicated by the parameter.

The orbit of a critical point is called a *critical orbit* and its importance is due to the following theorem.

Theorem 5.2. *If $f : \mathbb{C} \to \mathbb{C}$ has an attractive or super-attractive orbit, then that orbit must attract at least one critical point.*

Note that this is really a theorem of complex dynamics. There is an analogous statement for real dynamics but it's a bit more complicated and its proof takes us a bit farther astray than we want. This is a great example of complex analysis being, in some ways, more elegant than real analysis.

Regardless, the theorem has important implications for real iteration. For example, a polynomial of degree n can have at most $n-1$ attractive orbits. Furthermore, if all the critical points happen to be real, we can find all the attractive behavior by simply iterating from the critical points. If we do this systematically for the quadratic family, plotting the columns to generate the bifurcation diagram, we get figure 5.3

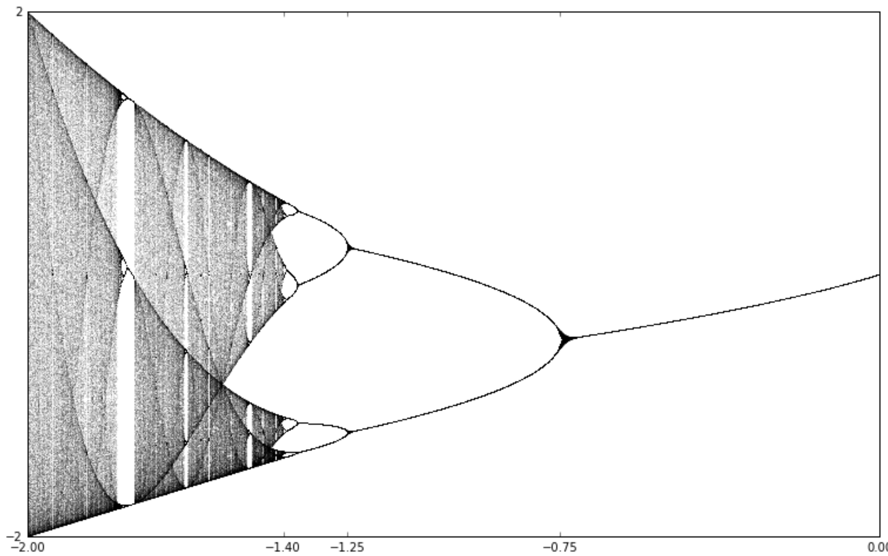

Figure 5.3: The bifurcation diagram for the quadratic family

We can interpret this diagram as follows:

- For $-0.75 < c < 0$, there is an attractive fixed point.
- For $-1.25 < c < -0.75$, there is an attractive orbit of period 2.
- As c passes from just above -0.75 to just below -0.75, the dynamics of f_c undergo a *bifurcation*.
- For c just a little less than -1.25, there is an attractive orbit of period four. This orbit bifurcates soon into an attractive orbit of period 8. It appears that this behavior continues as c decreases.
- For c somewhere around $c \approx -1.4$, the period doubling appears to stop and we get more complicated behavior.

Generally, a bifurcation occurs at a parameter value $c = c_0$ if the global dynamical behavior of the function f_c undergoes some qualitative change as c passes through c_0. There are number of different types of bifurcations that can occur, depending on the nature of the qualitative behavior under consideration. The bifurcations that are evident in 5.3 in the range $-1.4 < c < 0$ are called *period doubling bifurcations*.

5.2 The period doubling cascade

Let's work towards a deeper, theoretical understanding of the period doubling that we see in the bifurcation diagram of figure 5.3. Again, we are dealing with the family of functions $f_c(x) = x^2 + c$. For c just a bit larger than -0.75 it appears that we have an attractive fixed point while, for c just a bit smaller than -0.75, it appears that we have an attracting orbit of period two. Why, exactly does this happen?

First, let's explore the fixed points of f_c; we can find them by solving $f_c(x) = x$:

$$x^2 + c = x \iff x^2 - x + c = 0.$$

Applying the quadratic formula, we find

$$x = \frac{1 \pm \sqrt{1-4c}}{2}.$$

For $c < 1/4$, we have two real fixed points but a glance at the graphs from figure 5.1 shows that it's the smaller of these two fixed points we're interested in. Of course, $f'(x) = 2x$, so the value of the derivative at the smaller fixed point is $1 - \sqrt{1-4c}$. Plugging $c = -3/4$ into this formula, we find that this is -1. For c slightly larger than $-3/4$, this is bigger than -1 and for c slightly smaller than $-3/4$, this is smaller than -1. This explains why we have an attractive fixed point for c slightly larger than $-3/4$ that is no longer attractive once c passes below $-3/4$.

Now, we ask - why does the attractive orbit of period two appear as the attractive fixed point disappears? To see this, we consider the function

$$F_c(x) = f_c \circ f_c(x) = (x^2 + c)^2 + c = x^4 + 2cx^2 + (c^2 + c).$$

We are interested in the fixed points, thus we must solve

$$x^4 + 2cx^2 + (c^2 + c) = x \text{ or } x^4 + 2cx^2 - x + (c^2 + c) = 0. \tag{5.1}$$

Here is an observation that helps us factor this polynomial: Any point that is fixed by f_c must also be fixed by F_c. Thus, we expect $x^2 + c - x$ to be a factor of the polynomial in (5.1). Using this, we find that

$$x^4 + 2cx^2 - x + (c^2 + c) = (x^2 - x + c)(x^2 + x + c + 1).$$

We can then apply the quadratic formula to get the two new fixed points of F_c, namely

$$x = \frac{-1 \pm \sqrt{1 - 4(c+1)}}{2} = \frac{-1 \pm \sqrt{-(3+4c)}}{2}.$$

These two points form an orbit of period two for f_c. Since $f_c'(x) = 2x$ we can multiply those points by two and multiply the results to get the multiplier for the orbit. The result is:

$$(-1 + \sqrt{-(3+4c)})(-1 - \sqrt{-(3+4c)}) = 4 + 4c.$$

When $c = -3/4$, the multiplier is 1. For c a little less than $-3/4$, the multiplier is a little less than one. Hence the orbit has become attractive.

A nice way to visualize this is to plot f_c^2 together with f_c and $y = x$ on the same set of axes for a few different choices of c. This is shown in figure 5.4 where we can see exactly how The fixed point went from attractive to repulsive while an attractive orbit of period two showed up as c passed below -0.75.

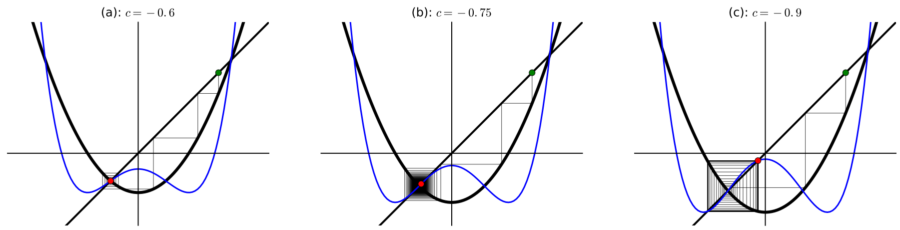

Figure 5.4: Bifurcation

6 Conjugacy

Figures 2.1 and 5.1 show that the iterative behavior of the logistic family and the quadratic family are very similar. In a sense, they are identical. We make that notion precise in this section.

Definition 6.1 (Conjugacy)**.** Let S and T be sets and suppose that $f : S \to S$ and $g : T \to T$. We say that f is *semi-conjugate* to g if there is a surjective function $\varphi : T \to S$ such that

$$f \circ \varphi = \varphi \circ g.$$

The function φ is called a *semi-conjugacy*. In the even that φ is bijective, then we say that f and g are conjugate.

Note that

$$f^2 \circ \varphi = f \circ f \circ \varphi = f \circ \varphi \circ g = \varphi \circ g \circ g = \varphi \circ g^2$$

and, by induction

$$f^n \circ \varphi = \varphi \circ g^n.$$

As a result, if (t_i) is an orbit of g, then $(\varphi(t_i))$ is an orbit of f.

Generally, the nicer φ is, the closer the relationship between the dynamics of f and the dynamics of g. If φ is bijective, then the relationship is quite close. If φ is continuous with continuous inverse, then topological properties of the orbits will be preserved. If S and T are sets of real or complex numbers and $\varphi(x) = ax + b$, then an orbit of one function will be geometrically similar to an orbit of the other. The dynamical systems are truly identical, up to a scaling.

Example 6.2. Show that $f(x) = x^2 - 1$ is conjugate to $g(x) = \frac{1}{2}x^2 + 2x - 2$ via the conjugacy $\varphi(x) = \frac{1}{2}x + 1$.

Solution. We simply compute

$$\begin{aligned} f(\varphi(x)) &= \left(\frac{1}{2}x + 1\right)^2 - 1 = \frac{1}{4}x^2 + x \\ \varphi(g(x)) &= \frac{1}{2}\left(\frac{1}{2}x^2 + 2x - 2\right) + 1 = \frac{1}{4}x^2 + x. \end{aligned}$$

Figure 6.3 illustrates the similarity between the two functions.

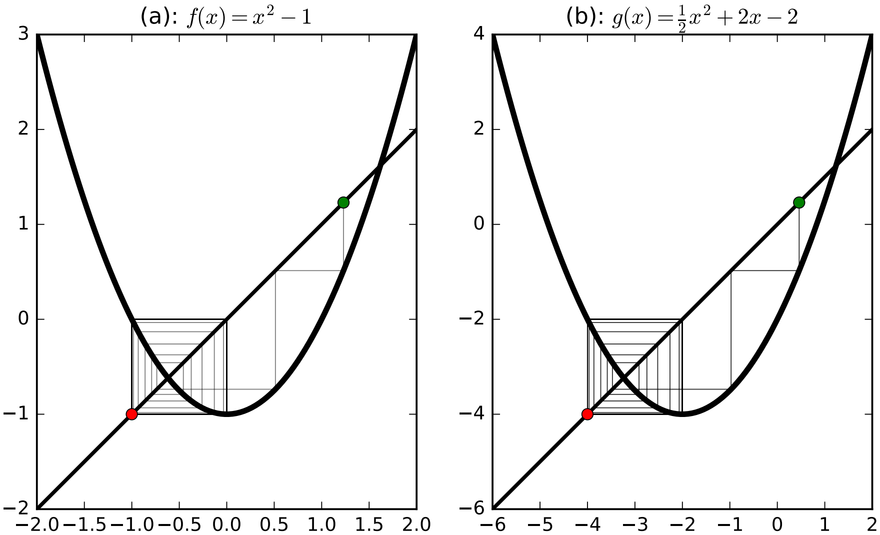

Figure 6.3: Cobweb plots for conjugate functions

If you suspect that f is conjugate to g via a conjugacy of the form $\varphi(x) = ax + b$, then you can find that conjugacy by setting $f(\varphi(x)) = \varphi(g(x))$. If you compare coefficients, you should get a system of equations that you can solve for a and b yielding the conjugacy.

Exercise 6.4. Find a conjugacy of the form $\varphi(x) = ax+b$ from $f(x) = x^2 - 2$ to $g(x) = 4x(1-x)$.

Exercise 6.4 can be generalized. In fact, the quadratic family for $-2 \leq c \leq 1/4$ is identical to the logistic family for $1 \leq \lambda \leq 4$.

Exercise 6.5. Show that $f(x) = x^2 + (2\lambda - \lambda^2)/4$ is conjugate to $g(x) = \lambda x(1-x)$ via the conjugacy $\varphi(x) = -\lambda x + \lambda/2$.

7 The doubling map and chaos

A glance at the cobweb plots of $f(x) = x^2 - 2$ and $g(x) = 4x(1-x)$ shows that they both exhibit very complicated behavior. In fact, they are chaotic in a perfectly quantitative sense. In this section, we'll introduce the doubling map, which is (in a sense) the prototypical chaotic map. After seeing why it's chaotic, we'll show that it's conjugate to $f(x) = x^2 - 2$, implying that it too is chaotic.

7.1 The doubling map

Let H denote the half-open, half-closed unit interval:

$$H = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}.$$

The doubling map d is the function $d : H \to H$ defined by

$$d(x) = 2x \bmod 1.$$

A graph of the doubling map together with a typical cobweb plot starting at an irrational number is shown in figure 7.1

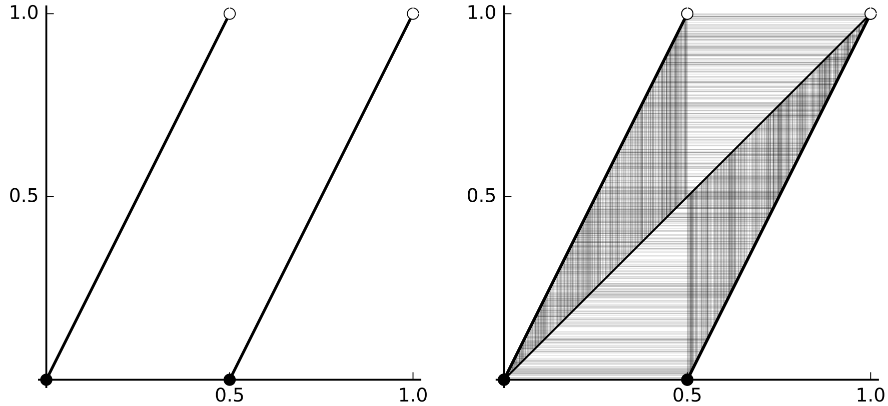

Figure 7.1: The doubling map

As it turns out, the doubling map is particularly easy to analyze if we consider its effect on the binary representation of a number. Suppose that $x \in H$ has binary representation

$$x = 0._2 b_1 b_2 b_3 b_4 b_5 \cdots ,$$

where each b_i is a zero or a one. (In computer parlance, the b_is are called the *bits* of the number.) Of course, some numbers have multiple binary representations. For example,

$$0._2 1 = 0._2 0\overline{1} = \frac{1}{2},$$

To ensure uniqueness, we agree to consistently represent numbers with a terminating binary expansion, if possible. Thus, representations that end with a repeating 1 are excluded.

Now, the effect of d on the binary representation of x is simple:

$$d(x) = 0._2 b_2 b_3 b_4 b_5 \cdots .$$

That is, the effect of d is to simply shift the bits of x to the left, discarding the bit that shifted into the ones place. This observation makes it very easy to find orbits with specific properties. Suppose, for example, we want an orbit of period 3. Simply pick (almost) any number of the form

$$x = 0._2 \overline{b_1 b_2 b_3}$$

The only caveat is that we can't have all b_is the same for that would lead to either zero (which is fixed) or one (which is not in H). As a concrete example,

$$x = 0._2 \overline{001} = \sum_{k=1}^{\infty} \frac{1}{8^k} = \frac{1}{7}$$

has period 3. In fact, it's easy to verify that

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{8}{7} = \frac{1}{7} \bmod 1$$

under the doubling map.

Another nice feature of this representation is that there is a simple correspondence between the binary expansion of a number and its position in the unit interval. Every number with a binary expansion starting with a zero lies in the left half of the unit interval, while every number starting with a one lies in the right half. The first two bits of a number specify in which quarter of the interval the number lies; the first three bits specify in which eighth of the unit interval the number lies, as shown in figure 7.2

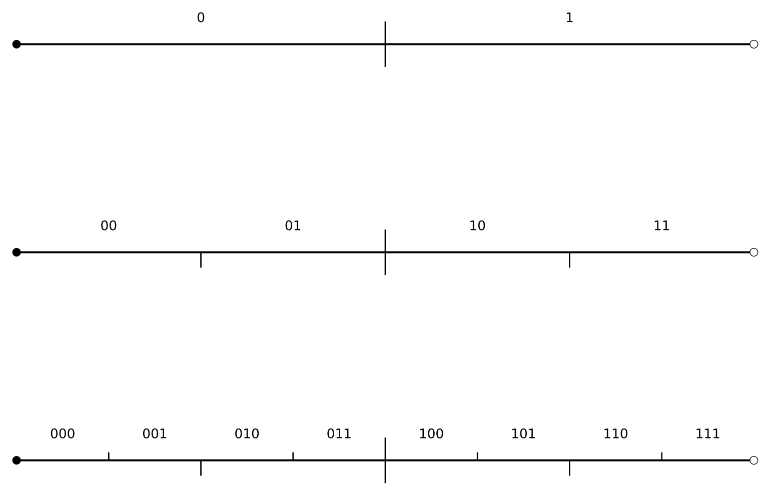

Figure 7.2: Dyadic intervals

More generally, given $n \in \mathbb{N}$, we can break the unit interval up into n pieces with length $1/2^n$ and endpoints $i/2^n$ for $i = 0, 1, \ldots, 2^n$. These are called *dyadic intervals* and their endpoints (number of the form $1/2^n$) are called dyadic rationals. The first n bits of a number specify in which n^{th} level dyadic interval that number lies. In fact, the left hand endpoint of a dyadic interval has a terminating binary expansion which tells you exactly the first n bits of all the points in that interval.

Now, suppose that

$$x_1 = 0_{\dot{2}} b_1 b_2 \cdots b_n b_{n+1} b'_{n+2} \cdots \quad \text{and} \quad x_2 = 0_{\dot{2}} b_1 b_2 \cdots b_n b'_{n+1} b'_{n+2} \cdots .$$

Thus, the binary expansions of x_1 and x_2 agree up to at least the n^{th} spot but potentially disagree after that. Then, our geometric understanding of dyadic intervals allows us to easily see that,

$$|x_1 - x_2| \leq \frac{1}{2^n}.$$

Of course, there's also a simple algebraic proof of this fact, based on the fact that the bits cancel for $k \leq n$

$$\begin{aligned} |x_1 - x_2| &= \left| \sum_{k=n+1}^{\infty} \frac{x_k - x'_k}{2^k} \right| \\ &\leq \sum_{k=n+1}^{\infty} \frac{|b_k - b'_k|}{2^k} \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}. \end{aligned}$$

7.2 Chaos

We can now prove three claims about the doubling map that, together, assert that the doubling map displays some of the essential features of chaos. First, we'll need to state and prove a lemma.

Lemma 7.3. *Suppose that*

$$x = 0_{\dot{2}} b_1 b_2 b_3 \cdots \quad \text{and} \quad y = 0_{\dot{2}} b_1' b_2' b_3' \cdots$$

are elements of H that satisfy $b_1 \neq b_1'$ but $b_2 = b_2'$. Then $|x - y| \geq 1/4$.

Proof. Computing the difference using the binary representations, taking into account that the terms disagree in the first spot and agree in the second, and finally applying the reverse triangle inequality, we get

$$\begin{aligned} |x-y| &= \left| \sum_{i=1}^{\infty} \frac{b_i - b_i'}{2^n} \right| = \left| \pm\frac{1}{2} + \sum_{i=3}^{\infty} \frac{b_i - b_i'}{2^n} \right| \\ &\geq \left| \left| \pm\frac{1}{2} \right| - \left| \sum_{i=3}^{\infty} \frac{b_i - b_i'}{2^n} \right| \right| \geq \left| \frac{1}{2} - \frac{1}{4} \right| = \frac{1}{4}. \end{aligned}$$

□

A geometric interpretation of this lemma is as follows. The fact that the two points disagree in the first spot means that they cannot lie in the same half of H. The fact that they do agree in the second spot means that they lie in the same quarter relative to their half, as shown in figure 7.4. Clearly, any two such points cannot be within 1/4 of one another.

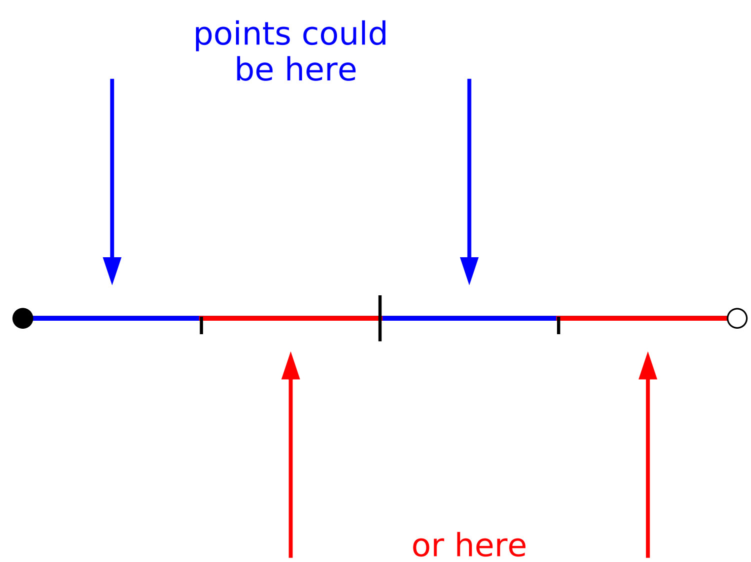

Figure 7.4: Possible positions of points in lemma 7.3

Claim 7.5 (Sensitive dependence on initial conditions)**.** *For every $x \in H$ and for every $\varepsilon > 0$, there is some $y \in H$ and an $n \in \mathbb{N}$ such that $|x - y| < \varepsilon$ yet $|d^n(x) - d^n(y) \geq 1/4$.*

Proof. Choose $n \in \mathbb{N}$ large enough so that $1/2^n < \varepsilon$. Now suppose that $x \in H$ has binary expansion

$$x = 0_{\dot{2}} b_1 b_2 \cdots b_n b_{n+1} b_{n+2} \cdots .$$

Define $y \in H$ so that

$$y = 0_{\dot{2}} b_1 b_2 \cdots b_n (1 - b_{n+1}) b_{n+2} \cdots .$$

That is, the bits of y agree with those of x in the first n spots, disagree with x in the $(n+1)^{\text{st}}$ spot, and finally agree with x again in the $(n+2)^{\text{nd}}$ spot.

Then, the numbers $d^n(x)$ and $d^n(y)$ satisfy the hypotheses of lemma 7.3, thus $|d^n(x) - d^n(y)| \geq 1/4$. ☐

Claim 7.6 (Denseness of periodic orbits)**.** *For every open interval $I \subset H$, there is some periodic orbit with an element in I.*

Proof. Let $x \in I$ and choose $n \in \mathbb{N}$ large enough so that

$$(x, x + 1/2^n) \subset I.$$

Now suppose that

$$x = 0_{\dot{2}} b_1 b_2 \cdots b_n \cdots .$$

Then,

$$\hat{x} = 0_{\dot{2}} \overline{b_1 b_2 \cdots b_n}$$

is a periodic point in I. ☐

Claim 7.7 (A dense orbit)**.** *There is a point $x \in H$ with the property that, for every open interval $I \subset H$, there is some iterate of x in I.*

Proof. We'll define x by specifying its binary expansion. We begin by writing down all possible *finite* binary strings:

$$0,\ 1,\ \ 00,\ 01,\ 10,\ 11,\ \ 000,\ 001,\ 010,\ 011,\ 100,\ 101,\ 110,\ 111, \ldots$$

We then concatenate these to obtain the binary representation of x

$$x = 0_{\dot{2}} 0\, 1\, 00\, 01\, 10\, 11\, 000\, 001\, 010\, 011\, 100\, 101\, 110\, 111 \ldots$$

Now, let $I \subset H$ be an open interval. We claim that there is some iterate of x in I. To see that, let L denote the length of I and choose $n \in \mathbb{N}$ large enough so that

$$\frac{1}{2^n} < \frac{1}{2} L.$$

Let i be the smallest integer such that $i/2^n \in I$. Note that we must also have $(i+1)/2^n \in I$. Thus, the dyadic interval $[i/2^n, (i+1)/2^n)$ is wholly contained in I and the first n bits of every point in that interval agree with $i/2^n$. So, let

$$\frac{i}{2^n} = 0_{\dot{2}} b_1 b_2 \cdots b_n$$

and note that, by construction, the string $b_1 b_2 \cdots b_n$ appears somewhere in the binary expansion of x. Thus, we can apply the doubling function to the point x some number, say m, times to obtain

$$d^m(x) = 0_{\dot{2}} b_1 b_2 \cdots b_n \cdots .$$

The number $d^m(x)$ is then an iterate of x that lies in I. ☐

While there is no truly universally accepted definition of chaos, claims 7.5, 7.6, and 7.7 are generally agreed to express some of the essential features of chaos.

7.3 A chaotic quadratic

Let $f(x) = x^2 - 2$. We now show that f is semi-conjugate to the doubling map d under the semi-conjugacy $\varphi(x) = 2\cos(2\pi x)$. As a result, φ maps all the orbit types that d has to an orbit of f with similar properties. Thus, f is chaotic.

Claim 7.8. *The map $f(x) = x^2 - 2$ is semi-conjugate to the doubling map $d(x) = 2x \bmod 1$ under the semi-conjugacy $\varphi(x) = 2\cos(2\pi x)$.*

Proof. We must simply show that $f \circ \varphi = \varphi \circ d$, so let's compute. First,

$$f(\varphi(x)) = 2(2\cos(2\pi x))^2 - 2 = 4\cos^2(2\pi x) - 2.$$

Well, that was easy. The next part is a little trickier - we just need to apply a couple of trig identities and use the fact that we can drop the mods inside the squared trig functions due to the symmetries of those functions.

$$\begin{aligned}
\varphi(d(x)) &= 2\cos(2\pi(2x \bmod 1)) \\
&= 2(\cos^2(\pi(2x \bmod 1)) - \sin^2(\pi(2x \bmod 1))) \\
&= 2(\cos^2(2\pi x) - \sin^2(2\pi x)) \\
&= 2(\cos^2(2\pi x) - (1 - \cos^2(2\pi x))) \\
&= 2(2\cos^2(2\pi x) - 1) \\
&= 4\cos^2(2\pi x) - 2
\end{aligned}$$

□

Example 7.9. Find a point of period 11 for the chaotic quadratic $f(x) = x^2 - 2$.

Solution. First, it's easy to find an orbit of period 11 for the doubling map. One example is

$$0_{\dot{2}}00000000001 = \sum k = 1^{\infty} \frac{1}{2^{11k}} = \frac{1}{2047}.$$

The point behind conjugacy is that $\varphi(1/2047) = 2\cos(2\pi/2047)$ will be a point of period 11 for f. The reader is advised to check this numerically!

8 A few notes on computation

Many of the results in these notes have been illustrated on the computer and some of the exercises require a computational approach. Whenever using the computer, it is always wise to examine the results critically. Here's a simple numerical example where things clearly go awry. In it, we are iterating the function $f(x) = x^2 - 9.1x + 1$ from the fixed point $x_0 = 0.1$. We should generate just a constant sequence.

```
x = 0.1
for i in range(20):
    x = x**2 - 9.1*x+ 1
    print(x)
```

```
# Output:
0.09999999999999998
0.1000000000000002
0.0999999999999982
0.10000000000001608
0.09999999999985698
0.10000000000127285
0.0999999999886717
0.10000000010082188
0.0999999991026852
0.10000000798610176
0.09999992892369436
0.10000063257912528
0.09999437004618517
0.1000501066206484
0.09955405358690272
0.10396912194476915
0.06469056862056699
0.41550069522129274
-2.608415498784386
31.540412453236513
```

Uh-oh!

It must be understood that this is a simple consequence of the nature of floating point arithmetic. Part of the issue is that the decimal number 0.1 or 1/10 is not exactly representable in binary. In fact,

$$\frac{1}{10} = 0_{\dot{2}}\overline{0011} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{3}{16^k}.$$

Thus, the computer *must* introduce round-off error in the computation. Furthermore, 0.1 is a repelling fixed point of the function. Thus, that round-off error is magnified with each iteration. Our study of dynamics has illuminated a critical issue in numerical computation!

In the terminology of numerical analysis, computation near an attractive fixed point is stable while computation near a repelling fixed point is unstable. Generally speaking, stable computation is trustworthy while unstable computation is not. The implication for the pictures that we see here is that illustration of attractive behavior should be just fine. In figure 2.1, for example, images (b) and (c) illustrate attraction to a fixed point that not only involves stable computation but also agrees with our theoretical development. We are happy with those figures. The cobweb plot shown in 2.1 (d), however, should frankly be viewed with some suspicion.

The same can be said for the bifurcation diagram in figure 5.3. In much of that image, we see a gray smear indicating chaos. How can we trust that? Well, first, theory tells us that there really *is* chaos. That is, there are orbits that are dense in some interval for many c values. Furthermore, much of the image shows attractive regions and we can be confident in that portion.

In fact, in many of the images that we will generate later - Julia sets, the Mandelbrot set, and similar images - the stable region dominates. Thus, we can be confident in overall image because the unstable region is the complement of the stable region. We might not be confident in computations involving some particular point, but we can be confident in the overall picture. (This will, perhaps, be more clear as we move into complex dynamics.)

Nonetheless, sometimes we want to experiment with genuinely unstable dynamics. One way to improve our confidence in these kinds of computations is to use high precision numbers. Consider, for example, the cobweb plot of the doubling map shown in figure 7.1. A naive approach to generate the first few terms of an orbit associated with the doubling map might be as follows:

```
import numpy as np
x = 1/np.pi
for i in range(55):
    x = 2*x%1
    print(x)
```

```
# Truncated output
0.636619772368
0.273239544735
0.54647908947
...
0.375
0.75
0.5
0.0
0.0
```

We've reached the fixed point zero and now we're stuck! Even if we iterate 1000 times, we'll generate a cobweb plot that looks like figure 8.1

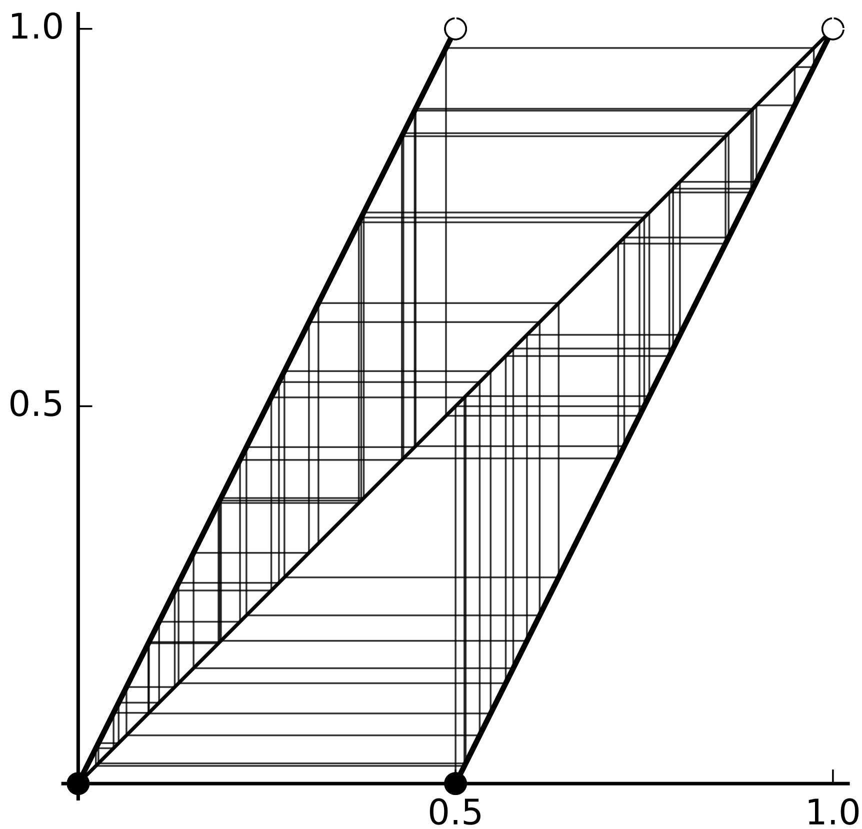

Figure 8.1: An inaccurate cobweb plot

The cobweb plot shown in figure 7.1 was generated using the mpmath multi-precision library for Python with code that looked something like so:

```
from mpmath import mp
mp.prec = 1000
x = 1/mp.pi
for i in range(1000):
    x = 2*x%1
    print(float(x))
```

```
# Truncated output
0.6366197723675813
0.27323954473516265
0.5464790894703253
...
0.375
0.75
0.5
0.0
0.0
```

While the truncated output looks the same, note that this was after 1000 iterates. This behavior makes perfect sense if you understand that the doubling map loses one bit of precision with every iterate.

9 Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. We say that x_0 is a simple root of f if $f(x_0) = 0$ and $f'(x_0) \neq 0$. Show that if x_0 is a simple root of f, then x_0 is a super-attracting fixed point of the Newton's method iteration function N for f.

2. Find an example of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ that attracts no critical point.

Hint. Draw a graph. Of course, you can't violate theorem 5.2.

3. Let $f(x) = x^2 - 4x + 5$. Show that f has a super-attractive orbit of period 2.

4. Let $f(x) = 3x^2 - 6x + 3.415$. Find all attractive orbits of f.

5. Find a value of c such that $f_c(x) = x^2 + c$ is affinely conjugate to $g(x) = (x-1)(x+2)$. Show that both functions have neutral fixed points.

6. Find an orbit of period 11 for the function $g(x) = 4x(1-x)$