Notes and HW on Green's and the Divergence Theorems

There are two big theorems at this stage of vector calculus - Green's Theorem and the Divergence Theorem. Here are some notes and problems on both of them.

The two theorems

First, recall Green's theorem. If C is a simple, closed, counter-clockwise oriented curve in the plane that encloses a region Ω and $\vec{F} = \langle P, Q \rangle$, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

The integral on the left is often written in the form $\oint_C P dx + Q dy$, but the form above allows easy connection with the divergence theorem, namely:

$$\oint_C \vec{F} \cdot d\vec{n} = \iint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA.$$

The integral with respect to the normal on the left measures flow (or flux) of the vector field across the curve. The expression $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ is called the *divergence* of the field $\vec{F}$ and is denoted $\text{div}\vec{F}$. Because of it's relationship with the flux integral, it is a local measure of how much material is flowing away (or diverging from) any particular point.

There's even a 3D version. Namely, if S is a closed, outwardly oriented surface in space (like a sphere or ellipsoid) that encloses a 3D domain Ω and $\vec{F} = \langle P, Q, R \rangle$, then

$$\iint_S \vec{F} \cdot d\vec{n} = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV.$$

Explanation of the divergence theorem

We can get a grip onto why the (2D) divergence theorem is true by taking a look at figure 1.

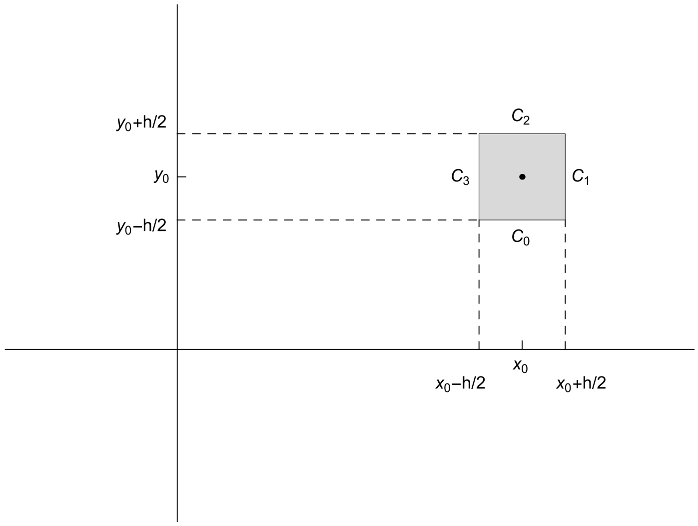

Figure 1: Examining the flow out of a small square yields the divergence theorem

The total flow of a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ out of the square is measured by

$$\oint_C \vec{F} \cdot d\vec{n}$$

and this can be broken into four parts along C_1, C_2, C_3, and C_4, each of which can be computed fairly easily. The flow across C_1, for example, is

$$\begin{aligned}\int_{C_1} \vec{F} \cdot d\vec{n} &= \int_{y_0-h/2}^{y_0+h/2} \langle P(x(t), y(t)), Q(x(t), y(t))\rangle \cdot \langle 1, 0\rangle dt \\ &= \int_{y_0-h/2}^{y_0+h/2} P(x_0 + h/2, t)dt \approx P(x_0 + h/2, y_0)h.\end{aligned}$$

Similarly, the flow across C_3 is

$$\int_{C_3} \vec{F} \cdot d\vec{n} \approx P(x_0 - h/2, y_0)h.$$

The difference of these gives us the net horizontal flow out of the domain, namely

$$P(x_0 + h/2, y_0)h - P(x_0 - h/2, y_0)h = \frac{P(x_0 + h/2, y_0) - P(x_0 - h/2, y_0)}{h} h^2 \approx \frac{\partial P}{\partial x}(x_0, y_0)h^2.$$

Similarly, the vertical flow out of the square is approximately $\dfrac{\partial Q}{\partial y}(x_0, y_0)h^2$ for small h. Adding these together, we get that the total flow out of the square is approximately

$$\left(\frac{\partial P}{\partial x}(x_0, y_0) + \frac{\partial Q}{\partial y}(x_0, y_0)\right) h^2.$$

Thus, the expression $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is telling us the local outward flow per unit area.

Problems

1. Let $\vec{F} = \langle 2x + 3y, -x + y \rangle$ and suppose that C is the positively oriented unit circle. Compute

$$\oint_C \vec{F} \cdot d\vec{r} \text{ and } \oint_C \vec{F} \cdot d\vec{n}$$

both directly and from the appropriate theorem.

2. Let $\vec{F} = \langle xy, -xy \rangle$ and suppose that C bounds the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 1$. Use Green's theorem and the divergence theorem to compute

$$\oint_C \vec{F} \cdot d\vec{r} \text{ and } \oint_C \vec{F} \cdot d\vec{n}.$$

3. Suppose that $\text{div}\vec{F} = 2x + 3y$. Find the approximate flux of a vector field across a circle of radius 0.1 centered at the point $(1, 1)$.

4. Suppose that $\vec{F}$ is a vector field in 3D and that $\text{div}\vec{F} = 2x + 3y + 4z$. Find the approximate flux of a vector field across a sphere of radius 0.1 centered at the point $(1, 1, 1)$.

5. Let $\vec{F}(x, y) = \langle x^2 + y, x - y^3 \rangle$ and let C denote the vertical line segment from $(2, 0)$ to $(2, 1)$. Evaluate

$$\int_C \vec{F} \cdot d\vec{n}$$

where the normal is oriented to the right.

6. Let C be the closed,piecewise curve figured by traveling in straight lines between the points $(-2, 1), (-2, -3), (1, -1), (1, 5)$ and back to $(-2, 1)$, in that order. Use Green's Theorem to evaluate the integral:

$$\oint_C (2xy)dx + (xy^2)dy.$$