4
The Integral

Previous chapters dealt with **differential calculus**. They started with the "simple" geometrical idea of the slope of a tangent line to a curve, developed it into a combination of theory about derivatives and their properties, examined techniques for calculating derivatives, and applied these concepts and techniques to real-life situations. This chapter begins the development of **integral calculus** and starts with the "simple" geometric idea of **area** — an idea that will spawn its own combination of theory, techniques and applications.

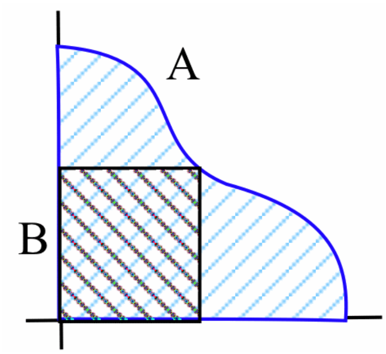
One of the most important results in mathematics, the Fundamental Theorem of Calculus, appears in this chapter. It unifies differential and integral calculus into a single grand structure. Historically, this unification marked the beginning of modern mathematics, and it provided important tools for the growth and development of the sciences.

The chapter begins with a look at area, some geometric properties of areas, and some applications. First we will examine ways of approximating the areas of regions such as tree leaves bounded by curved edges and the areas of regions bounded by graphs of functions. Then we will develop ways to calculate the areas of some of these regions exactly. Finally, we will explore the rich variety of uses of "areas."

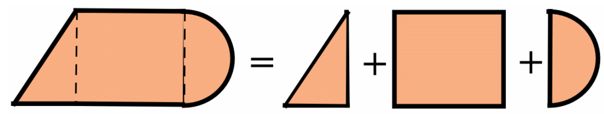
4.0 Area

The primary purpose of this introductory section is to help develop your intuition about areas and your ability to reason using geometric arguments about area. This type of reasoning will appear often in the rest of this book and is very helpful for applying the ideas of calculus.

The basic shape we will use is the rectangle: the area of a rectangle is (base) $\cdot$ (height). If the units for each side of the rectangle are "meters," then the area will have units (meters) $\cdot$ (meters) $=$ "square meters" $= \text{m}^2$. The only other area formulas needed for this section are for triangles (area $= \frac{1}{2} b \cdot h$) and for circles (area $= \pi \cdot r^2$). In addition, we will use (and assume to be true) three other familiar properties of area:

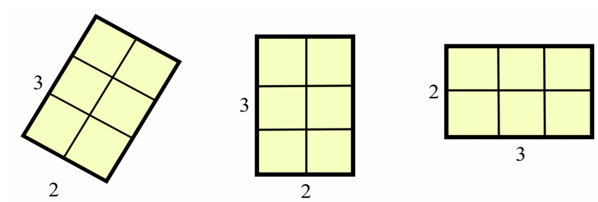


- **Addition Property**: The total area of a region is the sum of the areas of the non-overlapping pieces that comprise the region:

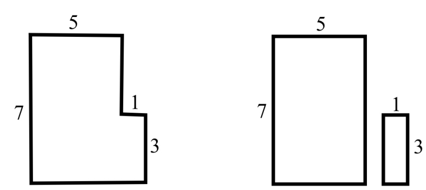


- **Inclusion Property**: If region B is inside region A (see margin), then the area of region B is less than or equal to the area of region A.

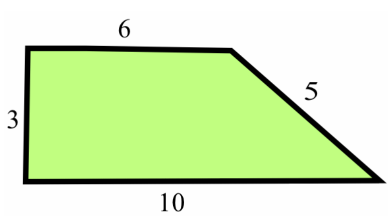
- **Location-Independence Property**: The area of a region does not depend on its location:



Example 1. Determine the area of the region shown below left.

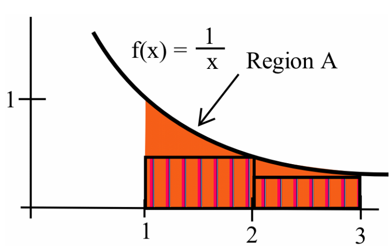


Solution. We can easily break the region into two rectangles (shown above right), with areas of 35 square inches and 3 square inches respectively, so the area of the original region is 38 square inches. ◀



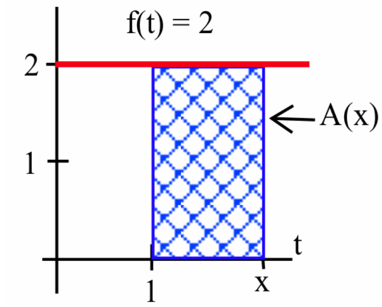
Practice 1. Determine the area of the trapezoidal region shown in the margin by cutting it in two ways: (a) into a rectangle and triangle and (b) into two triangles.

We can use our three area properties to deduce information about areas that are difficult to calculate exactly. Let A be the region bounded by the graph of $f(x) = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 3$. Because the two rectangles in the margin figure sit inside region A and do not overlap each other, the area of the rectangles, $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, is less than the area of region A.

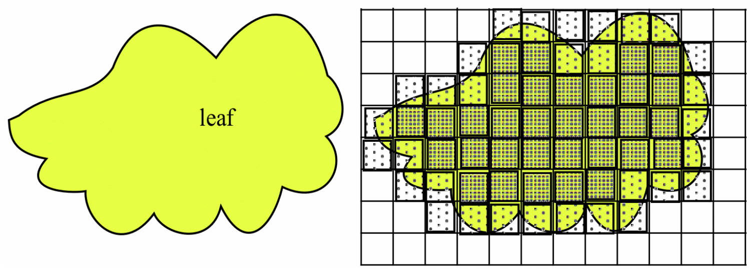


Practice 2. Build two rectangles, each with base 1 unit, with boundaries that extend outside the shaded region in the margin figure and use their areas to make another valid statement about the area of region A.

Practice 3. What can you say about the area of region A if we use "inside" and "outside" rectangles each with base $\frac{1}{2}$ unit?

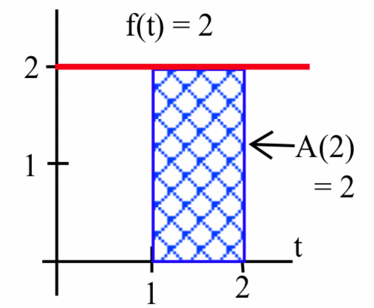


Example 2. The figure below right includes 32 dark squares, each 1 centimeter on a side, and 31 lighter squares of the same size. We can be sure that the area of the leaf below left is smaller than what number?

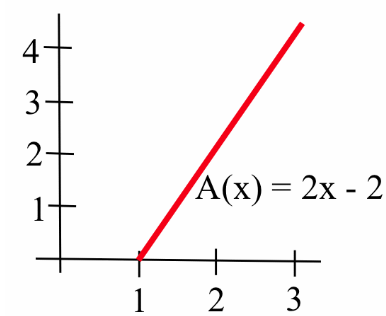


Solution. The area of the leaf is smaller than $32 + 31 = 63\,\text{cm}^2$. ◀

Practice 4. We can be sure that the area of the leaf is at least how large?

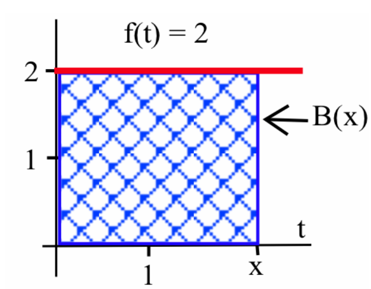


Functions can be defined in terms of areas. For the constant function $f(t) = 2$, define $A(x)$ to be the area of the rectangular region (top margin figure) bounded by the graph of f, the t-axis, and the vertical lines at $t = 1$ and $t = x$; we can easily see that $A(2) = 2$ (shaded region in the second margin figure). Similarly, $A(3) = 4$ and $A(4) = 6$. In general, $A(x) = (\text{base})(\text{height}) = (x-1)(2) = 2x - 2$ for any $x \geq 1$. From the graph of $y = A(x)$ (in the third margin figure) we can see that $A'(x) = 2$ for every value of $x > 1$.



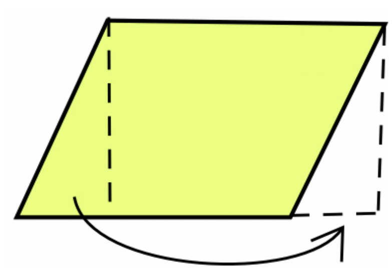
(The fact that $A'(x) = f(x)$ in the preceding discussion is not a coincidence, as we shall soon learn.)

Practice 5. For $f(t) = 2$, define $B(x)$ to be the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = 0$ and $t = x$ (see below left). Fill in the table below with the requested values of B. How are the graphs of $y = A(x)$ and $y = B(x)$ related?

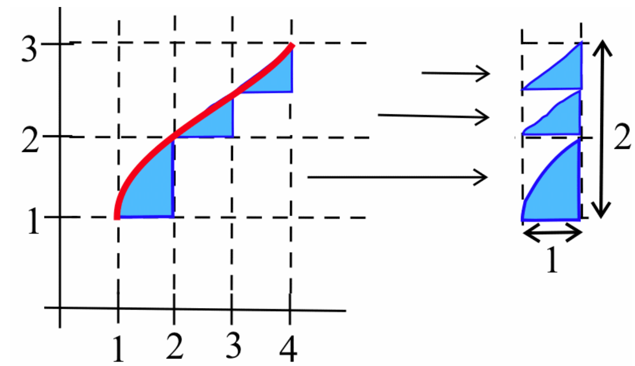


x	$B(x)$
0	
0.5	
1	
2	

Sometimes it is useful to move regions around. The area of a parallelogram is obvious if we move the triangular region from one side of the parallelogram to fill the region on the other side, resulting in with a rectangle (see margin).

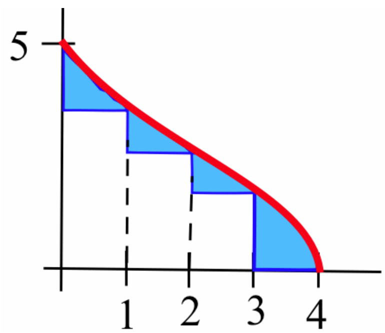


At first glance, it is difficult to estimate the total area of the shaded regions shown below left:



but if we slide all of them into a single column (above right), then becomes easy to determine that the shaded area is less than the area of the enclosing rectangle = (base)(height) = (1)(2) = 2.

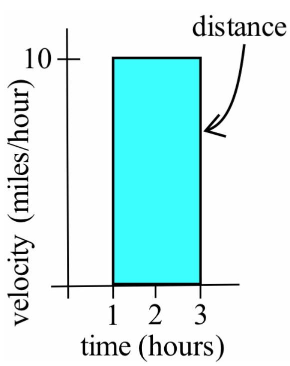
Practice 6. The total area of the shaded regions in the margin figure is less than what number?



Some Applications of "Area"

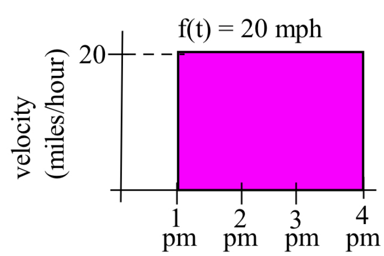
One reason "areas" are so useful is that they can represent quantities other than sizes of simple geometric shapes. For example, if the units of the base of a rectangle are "hours" and the units of the height are "$\frac{\text{miles}}{\text{hour}}$," then the units of the "area" of the rectangle are:

$$(\text{hours}) \cdot \left(\frac{\text{miles}}{\text{hour}}\right) = \text{miles}$$



a measure of distance. Similarly, if the base units are "pounds" and the height units are "feet," then the "area" units are "foot-pounds," a measure of work.

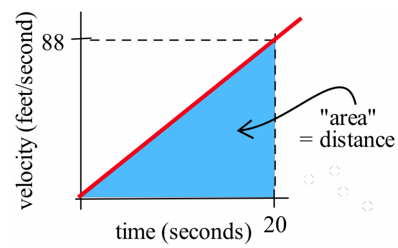
In the bottom margin figure, $f(t)$ is the velocity of a car in "miles per hour," and t is the time in "hours." So the shaded "area" will be $(\text{base}) \cdot (\text{height}) = (3\,\text{hours}) \cdot \left(20\,\frac{\text{miles}}{\text{hour}}\right) = 60$ miles, the distance traveled by the car in the 3 hours from 1:00 p.m. until 4:00 p.m.



Distance as an "Area"
If $f(t)$ is the (positive) forward velocity of an object at time t, then the "area" between the graph of f and the t-axis and the vertical lines at times $t = a$ and $t = b$ will equal the distance that the object has moved forward between times a and b.

This "area as distance" concept can make some difficult distance problems much easier.

Example 3. A car starts from rest (velocity $= 0$) and steadily speeds up so that 20 seconds later its speed is 88 feet per second (60 miles per hour). How far did the car travel during those 20 seconds?



Solution. We could answer the question using the techniques of Chapter 3 (try this). But if "steadily" means that the velocity increases linearly, then it is easier to use the margin figure and the concept of "area as distance."

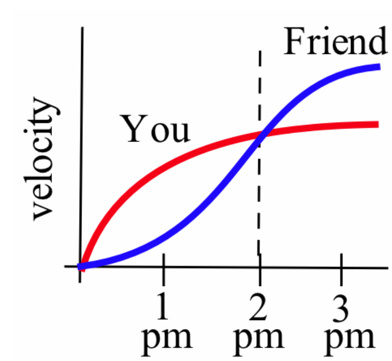
The "area" of the triangular region represents the distance traveled:

$$\text{distance} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(20\,\text{sec})\left(88\,\frac{\text{ft}}{\text{sec}}\right) = 880\,\text{ft}$$

The car travels a total of 880 feet during those 20 seconds. ◀

Practice 7. A train initially traveling at 45 miles per hour (66 feet per second) takes 60 seconds to decelerate to a complete stop. If the train slowed down at a steady rate (the velocity decreased linearly), how many feet did the train travel before coming to a stop?

Practice 8. You and a friend start off at noon and walk in the same direction along the same path at the rates shown in the margin figure.

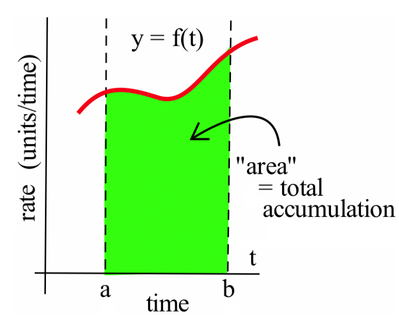


(a) Who is walking faster at 2:00 p.m.? Who is ahead at 2:00 p.m.?

(b) Who is walking faster at 3:00 p.m.? Who is ahead at 3:00 p.m.?

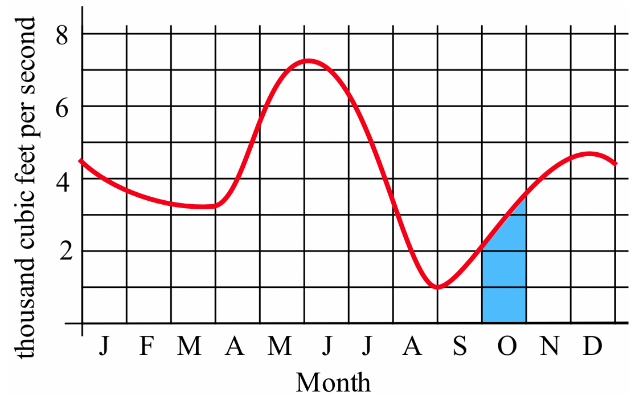
(c) When will you and your friend be together? (Answer in words.)

In the preceding Example and Practice problems, a function represented a rate of travel (in miles per hour, for instance) and the area represented the total distance traveled. For functions representing other rates, such as the production of a factory (bicycles per day) or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute) or the spread of a disease (newly sick people per week), the area will still represent the total amount of something.



"Area" as a Total Accumulation
If $f(t)$ represents a positive rate (in units per time interval) at time t, then the "area" between the graph of f and the t-axis and the vertical lines at times $t = a$ and $t = b$ will be the total amount of {something} that accumulates between times a and b (see margin).

For example, the figure at the top of the next page shows the flow rate (in cubic feet per second) of water in the Skykomish River near the town of Gold Bar, Washington. The area of the shaded region represents the total volume (cubic feet) of water flowing past the town during the month of October:

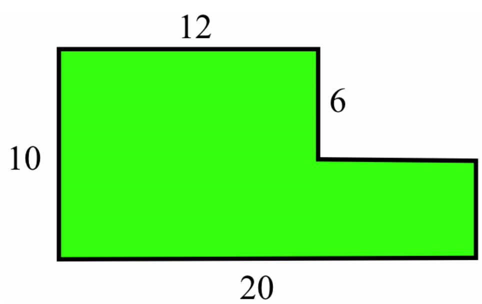


$$\begin{aligned}\text{total water} &= \text{"area"} = \text{area of rectangle} + \text{area of triangle}\\ &\approx \left(2000\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days}) + \frac{1}{2}\left(1500\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days})\\ &= \left(2750\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days}) = \left(2750\,\frac{\text{ft}^3}{\text{sec}}\right)(2592000\,\text{sec})\\ &\approx 7.128\times 10^9\,\text{ft}^3\end{aligned}$$

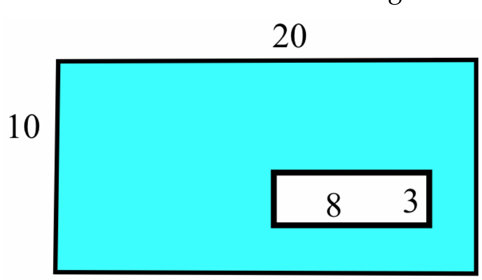
For comparison, the flow over Niagara Falls is about $2.12 \times 10^5\,\frac{\text{ft}^3}{\text{sec}}$.

4.0 Problems

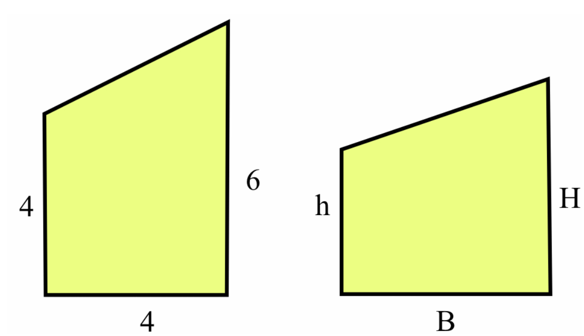
1. (a) Calculate the area of the shaded region:



   (b) Calculate the area of the shaded region:

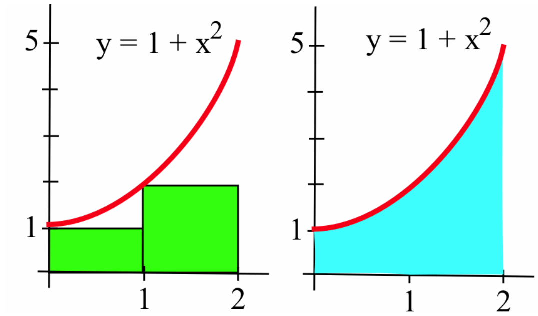


2. Calculate the area of the trapezoidal region in the figure below left by breaking it into a triangle and a rectangle.



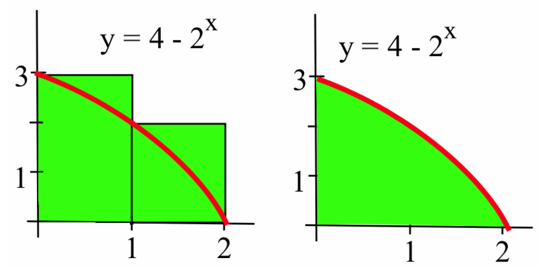
3. Break the region shown above right into a triangle and rectangle and verify that the total area of the trapezoid is $b \cdot \left(\frac{h+H}{2}\right)$.

4. (a) Calculate the sum of the rectangular areas in the region shown below left.



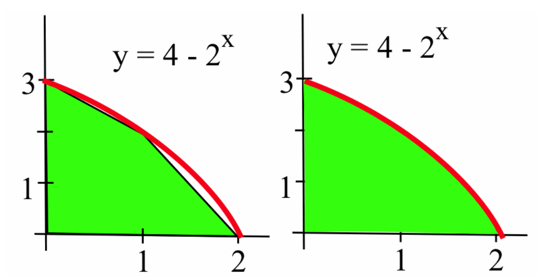
(b) What can you say about the area of the shaded region shown above right?

5. (a) Calculate the sum of the areas of the rectangles shown below left.



(b) What can you say about the area of the shaded region shown above right?

6. (a) Calculate the sum of the areas of the trapezoids shown below left.

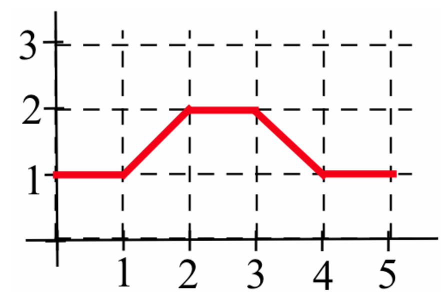


(b) What can you say about the area of the shaded region shown above right?

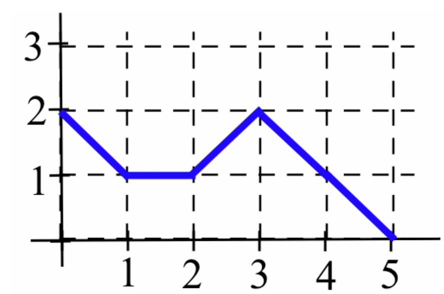
7. Consider the region bounded by the graph of $y = 2 + x^3$, the positive x-axis, the positive y-axis and the line $x = 2$. Use two well-placed rectangles to estimate the area of this region.

8. Consider the region bounded by the graph of $y = 9 - 3^x$, the positive x-axis and the positive y-axis. Use two well-placed trapezoids to estimate the area of this region.

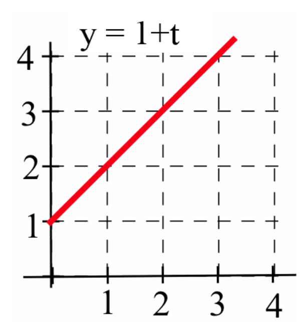
9. Let $A(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $A(x)$ for $x = 1, 2, 3, 4$ and 5.



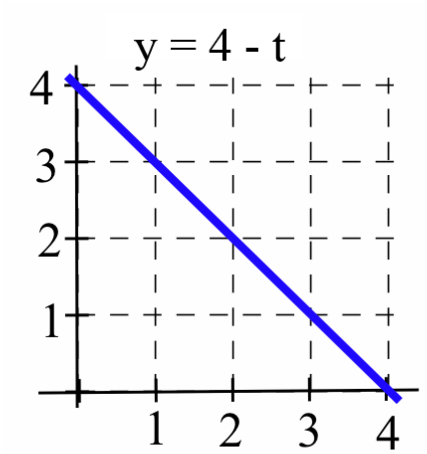
10. Let $B(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $B(x)$ for $x = 1, 2, 3, 4$ and 5.



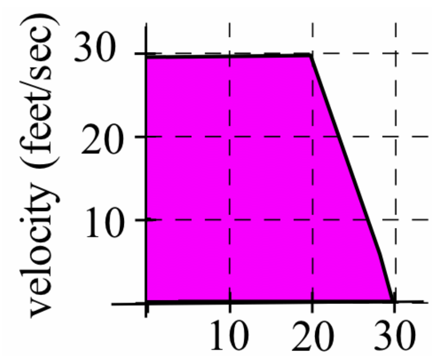
11. Let $C(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $C(x)$ for $x = 1, 2$ and 3, and use that information to deduce a formula for $C(x)$.



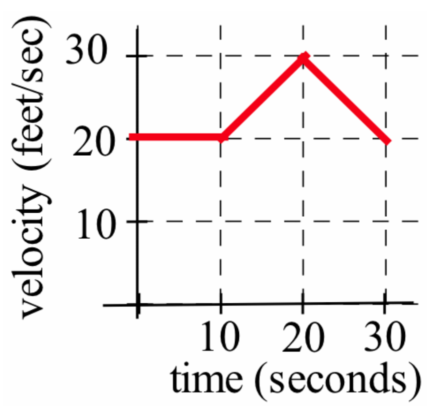
12. Let $A(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $A(x)$ for $x = 1$, 2 and 3, and find a formula for $A(x)$.



13. The figure below shows the velocity of a car during a 30-second time frame. How far did the car travel between $t = 0$ to $t = 30$ seconds?



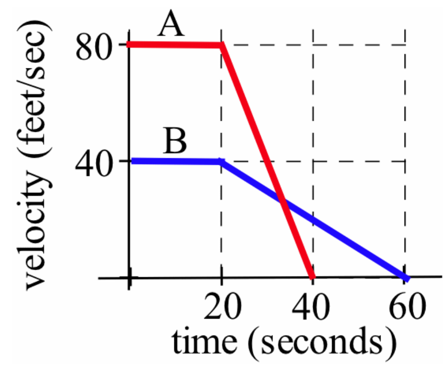
14. The figure below shows the velocity of a car during a 30-second time frame. How far did the car travel between $t = 0$ to $t = 30$ seconds?



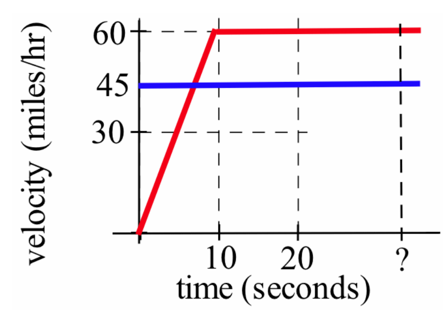
15. The figure below shows the velocities of two cars. From the time the brakes were applied:

(a) how long did it take each car to stop?

(b) which car traveled farther before stopping?



16. A speeder traveling 45 miles per hour (in a 25-mph zone) passes a stopped police car, which immediately takes off after the speeder. If the police car speeds up steadily to 60 mph over a 10-second interval and then travels at a constant 60 mph, how long—and how far—will it be before the police car catches the speeder, who continued traveling at 45 mph? (See figure below.)



17. Fill in the table with the units for "area" of a rectangle with the given base and height units.

base	**height**	**"area"**
miles per second	seconds	
hours	dollars per hour	
square feet	feet	
kilowatts	hours	
houses	people per house	
meals	meals	

4.0 Practice Answers

1. (a) $3(6) + \frac{1}{2}(4)(3) = 24$ (b) $\frac{1}{2}(3)(10) + \frac{1}{2}(6)(3) = 24$

2. outside rectangular area $= (1)(1) + (1)\left(\frac{1}{2}\right) = 1.5$

3. Using rectangles with base $= \frac{1}{2}$:

$$\text{inside area} = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3}\right) = \frac{57}{60} \approx 0.95$$
$$\text{outside area} = \frac{1}{2}\left(1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right) = \frac{72}{60} = 1.2$$

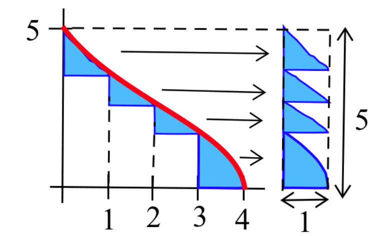
so the area of the region is between 0.95 and 1.2.

4. The leaf's area is larger than the area of the dark rectangles, $32\,\text{cm}^2$.

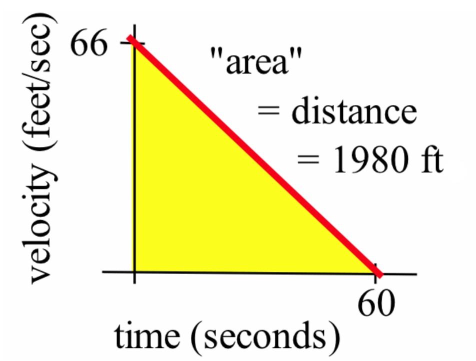
5. $y = B(x) = 2x$ is a line with slope 2, so it is parallel to the line $y = A(x) = 2x - 2$; see margin for table.

x	$B(x)$
0	0
0.5	1
1	2
2	4

6. Area $<$ area of the rectangle enclosing the shifted regions $= 5$; see margin figure.



7. Draw a graph of the velocity function:



and then use the concept of "area as distance":

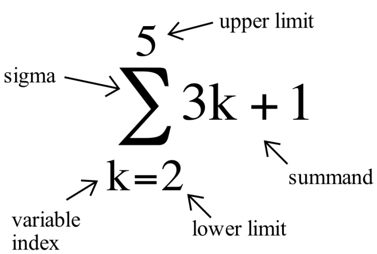
$$\begin{aligned}\text{distance} &= \text{area of shaded region} \\ &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(60\,\text{sec})\left(66\,\frac{\text{ft}}{\text{sec}}\right) = 1980\,\text{feet}\end{aligned}$$

8. (a) At 2:00 p.m. both are walking at the same velocity. You are ahead.
   (b) At 3:00 p.m. your friend is walking faster than you, but you are still ahead. (The "area" under your velocity curve is larger than the "area" under your friend's.)
   (c) You and your friend will be together on the trail when the "areas" (distances) under the two velocity graphs are equal.

4.1 Sigma Notation and Riemann Sums

One strategy for calculating the area of a region is to cut the region into simple shapes, calculate the area of each simple shape, and then add these smaller areas together to get the area of the whole region. When you use this approach with many sub-regions, it will be useful to have a notation for adding many values together: the sigma (Σ) notation.

summation	sigma notation	how to read it
$1^2+2^2+3^2+4^2+5^2$	$\sum_{k=1}^{5} k^2$	the sum of k squared, from k equals 1 to k equals 5
$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$	$\sum_{k=3}^{7} \frac{1}{k}$	the sum of 1 divided by k, from k equals 3 to k equals 7
$2^0+2^1+2^2+2^3+2^4+2^5$	$\sum_{j=0}^{5} 2^j$	the sum of 2 to the j-th power, from j equals 0 to j equals 5
$a_2+a_3+a_4+a_5+a_6+a_7$	$\sum_{n=2}^{7} a_n$	the sum of a sub n, from n equals 2 to n equals 7



The variable (typically i, j, k, m or n) used in the summation is called the **counter** or **index** variable. The function to the right of the sigma is called the **summand**, while the numbers below and above the sigma are called the **lower** and **upper limits** of the summation.

Practice 1. Write the summation denoted by each of the following:

$$\text{(a)} \sum_{k=1}^{5} k^3 \qquad \text{(b)} \sum_{j=2}^{7} (-1)^j \frac{1}{j} \qquad \text{(c)} \sum_{m=0}^{4} (2m+1)$$

In practice, we frequently use sigma notation together with the standard function notation:

$$\sum_{k=1}^{3} f(k+2) = f(1+2)+f(2+2)+f(3+2)$$
$$= f(3)+f(4)+f(5)$$
$$\sum_{j=1}^{4} f(x_j) = f(x_1)+f(x_2)+f(x_3)+f(x_4)$$

x	$f(x)$	$g(x)$	$h(x)$
1	2	4	3
2	3	1	3
3	1	-2	3
4	0	3	3
5	3	5	3

Example 1. Use the table to evaluate $\sum_{k=2}^{5} 2 \cdot f(k)$ and $\sum_{j=3}^{5} [5+f(j-2)]$.

Solution. Writing out the sum and using the table values:

$$\sum_{k=2}^{5} 2\cdot f(k) = 2\cdot f(2)+2\cdot f(3)+2\cdot f(4)+2\cdot f(5)$$
$$= 2\cdot 3+2\cdot 1+2\cdot 0+2\cdot 3 = 14$$

while:

$$\sum_{j=3}^{5}[5+f(j-2)] = [5+f(3-2)]+[5+f(4-2)]+[5+f(5-2)]$$
$$= [5+f(1)]+[5+f(2)]+[5+f(3)]$$
$$= [5+2]+[5+3]+[5+1]$$

which adds up to 21. ◀

Practice 2. Use the values in the preceding margin table to evaluate:

(a) $\sum_{k=2}^{5} g(k)$ (b) $\sum_{j=1}^{4} h(j)$ (c) $\sum_{k=3}^{5} [g(k)+f(k-1)]$

Example 2. For $f(x) = x^2+1$, evaluate $\sum_{k=0}^{3} f(k)$.

Solution. Writing out the sum and using the function values:

$$\sum_{k=0}^{3} f(k) = f(0)+f(1)+f(2)+f(3)$$
$$= \left(0^2+1\right)+\left(1^2+1\right)+\left(2^2+1\right)+\left(3^2+1\right)$$
$$= 1+2+5+10$$

which adds up to 18. ◀

Practice 3. For $g(x) = \frac{1}{x}$, evaluate $\sum_{k=2}^{4} g(k)$ and $\sum_{k=1}^{3} g(k+1)$.

The summand need not contain the index variable explicitly: you can write a sum from $k=2$ to $k=4$ of the constant function $f(k)=5$ as $\sum_{k=2}^{4} f(k)$ or $\sum_{k=2}^{4} 5 = 5+5+5 = 3\cdot 5 = 15$. Similarly:

$$\sum_{k=3}^{7} 2 = 2+2+2+2+2 = 5\cdot 2 = 10$$

Because the sigma notation is simply a notation for addition, it possesses all of the familiar properties of addition.

Summation Properties:

Sum of Constants: $\sum_{k=1}^{n} C = C+C+C+\cdots+C = n\cdot C$

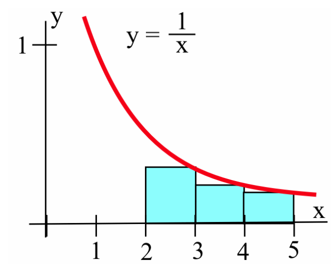
Addition: $\sum_{k=1}^{n} (a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Subtraction: $\sum_{k=1}^{n} (a_k-b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Constant Multiple: $\sum_{k=1}^{n} C\cdot a_k = C\cdot\sum_{k=1}^{n} a_k$

Problems 16 and 17 illustrate that similar patterns for sums of products and quotients are *not* valid.

Sums of Areas of Rectangles

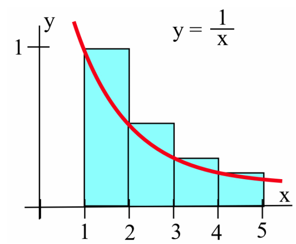


In Section 4.2, we will approximate areas under curves by building rectangles as high as the curve, calculating the area of each rectangle, and then adding the rectangular areas together.

Example 3. Evaluate the sum of the rectangular areas in the margin figure, then write the sum using sigma notation.

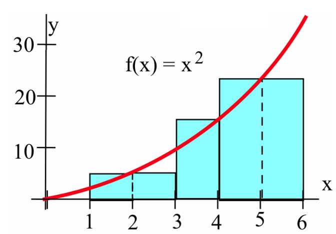
Solution. The sum of the rectangular areas is equal to the sum of (base) · (height) for each rectangle:

$$(1)\left(\frac{1}{3}\right)+(1)\left(\frac{1}{4}\right)+(1)\left(\frac{1}{5}\right)=\frac{47}{60}$$



which we can rewrite as $\sum_{k=3}^{5}\frac{1}{k}$ using sigma notation. ◀

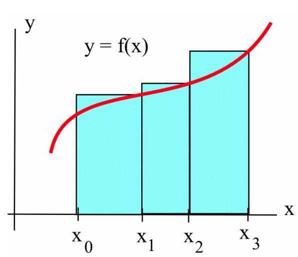
Practice 4. Evaluate the sum of the rectangular areas in the margin figure, then write the sum using sigma notation.



The bases of these rectangles need not be equal. For the rectangular areas associated with $f(x)=x^2$ in the margin figure:

rectangle	base	height	area
1	$3-1=2$	$f(2)=4$	$2\cdot4=8$
2	$4-3=1$	$f(4)=16$	$1\cdot16=16$
3	$6-4=2$	$f(5)=25$	$2\cdot25=50$

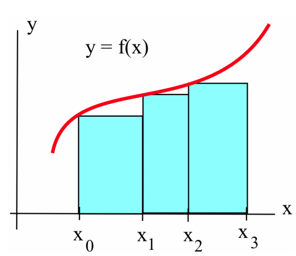
so the sum of the rectangular areas is $8+16+50=74$.



Example 4. Write the sum of the areas of the rectangles in the margin figure using sigma notation.

Solution. The area of each rectangle is (base) · (height):

rectangle	base	height	area
1	x_1-x_0	$f(x_1)$	$(x_1-x_0)\cdot f(x_1)$
2	x_2-x_1	$f(x_2)$	$(x_2-x_1)\cdot f(x_2)$
3	x_3-x_2	$f(x_3)$	$(x_3-x_2)\cdot f(x_3)$

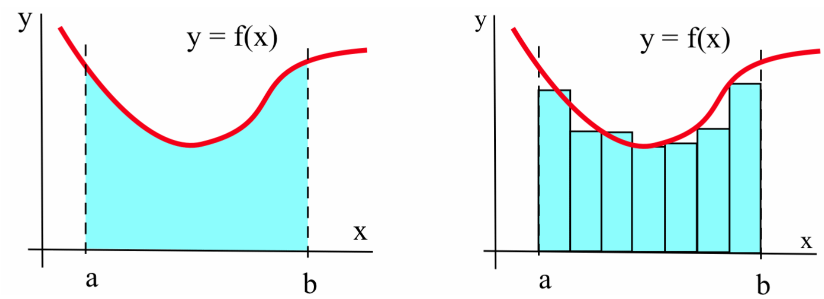


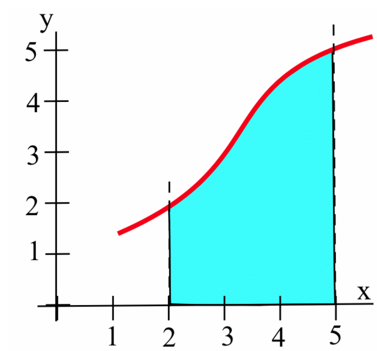
The area of the k-th rectangle is $(x_k-x_{k-1})\cdot f(x_k)$, so we can express the total area of the three rectangles as $\sum_{k=1}^{3}(x_k-x_{k-1})\cdot f(x_k)$. ◀

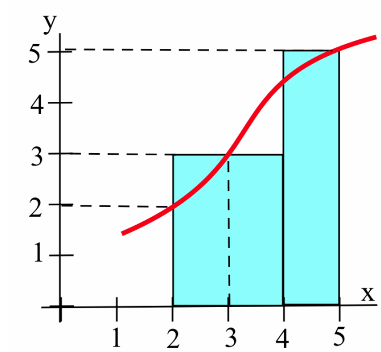
Practice 5. Write the sum of the areas of the shaded rectangles in the margin figure using sigma notation.

Area Under a Curve: Riemann Sums

Suppose we want to calculate the area between the graph of a positive function f and the x-axis on the interval $[a, b]$ (see below left).







One method to approximate the area involves building several rectangles with bases on the x-axis spanning the interval $[a, b]$ and with sides that reach up to the graph of f (above right). We then compute the areas of the rectangles and add them up to get a number called a **Riemann sum** of f on $[a, b]$. The area of the region formed by the rectangles provides an approximation of the area we want to compute.

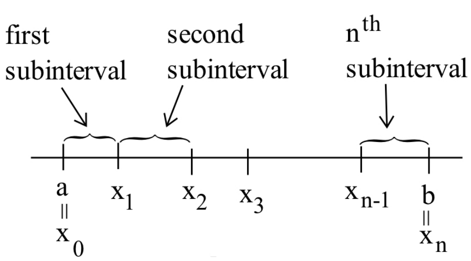
Example 5. Approximate the area shown in the margin between the graph of f and the x-axis spanning the interval $[2, 5]$ by summing the areas of the rectangles shown in the lower margin figure.

Solution. The total area is $(2)(3) + (1)(5) = 11$ square units. ◀

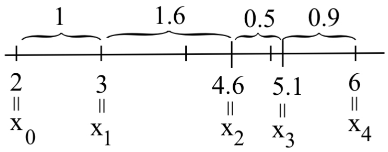
In order to effectively describe this process, some new vocabulary is helpful: a **partition** of an interval and the **mesh** of a partition.

A **partition** $\mathcal{P}$ of a closed interval $[a, b]$ into n subintervals consists of a set of $n + 1$ points $\{x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ listed in increasing order, so that $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$. (A partition is merely a collection of points on the horizontal axis, unrelated to the function f in any way.)

The points of the partition $\mathcal{P}$ divide $[a, b]$ into n subintervals:



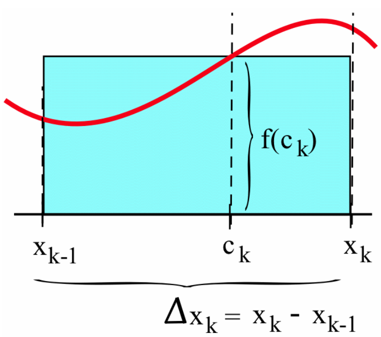
These intervals are $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, ..., $[x_{n-1}, x_n]$ with lengths $\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1$, $\Delta x_3 = x_3 - x_2$, ..., $\Delta x_n = x_n - x_{n-1}$. The points x_k of the partition $\mathcal{P}$ mark the locations of the vertical lines for the sides of the rectangles, and the bases of the rectangles have



lengths Δx_k for $k = 1, 2, 3, \ldots, n$. The **mesh** or **norm** of a partition $\mathcal{P}$ is the length of the longest of the subintervals $[x_{k-1}, x_k]$ or, equivalently, the maximum value of Δx_k for $k = 1, 2, 3, \ldots, n$.

For example, the set $\mathcal{P} = \{2, 3, 4.6, 5.1, 6\}$ is a partition of the interval $[2, 6]$ (see margin) that divides the interval $[2, 6]$ into four subintervals with lengths $\Delta x_1 = 1$, $\Delta x_2 = 1.6$, $\Delta x_3 = 0.5$ and $\Delta x_4 = 0.9$, so the mesh of this partition is 1.6, the maximum of the lengths of the subintervals. (If the mesh of a partition is "small," then the length of each one of the subintervals is the same or smaller.)

Practice 6. $\mathcal{P} = \{3, 3.8, 4.8, 5.3, 6.5, 7, 8\}$ is a partition of what interval? How many subintervals does it create? What is the mesh of the partition? What are the values of x_2 and Δx_2?

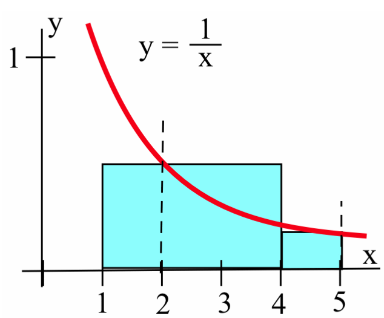


A function, a partition and a point chosen from each subinterval determine a **Riemann sum**. Suppose f is a positive function on the interval $[a, b]$ (so that $f(x) > 0$ when $a \le x \le b$), $\mathcal{P} = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ is a partition of $[a, b]$, and c_k is an x-value chosen from the k-th subinterval $[x_{k-1}, x_k]$ (so $x_{k-1} \le c_k \le x_k$). Then the area of the k-th rectangle is:

$$f(c_k) \cdot (x_k - x_{k-1}) = f(c_k) \cdot \Delta x_k$$

Definition:
A summation of the form $\displaystyle\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called a **Riemann sum** of f for the partition $\mathcal{P}$ and the chosen points $\{c_1, c_2, \ldots, c_n\}$.

This Riemann sum is the total of the areas of the rectangular regions and provides an approximation of the area between the graph of f and the x-axis on the interval $[a, b]$.



Example 6. Find the Riemann sum for $f(x) = \frac{1}{x}$ using the partition $\{1, 4, 5\}$ and the values $c_1 = 2$ and $c_2 = 5$ (see margin).

Solution. The two subintervals are $[1, 4]$ and $[4, 5]$, hence $\Delta x_1 = 3$ and $\Delta x_2 = 1$. So the Riemann sum for this partition is:

$$\sum_{k=1}^{2} f(c_k) \cdot \Delta x_k = f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2$$
$$= f(2) \cdot 3 + f(5) \cdot 1 = \frac{1}{2} \cdot 3 + \frac{1}{5} \cdot 1 = \frac{17}{10}$$

The value of the Riemann sum is 1.7. ◀

Practice 7. Calculate the Riemann sum for $f(x) = \frac{1}{x}$ on the partition $\{1, 4, 5\}$ using the chosen values $c_1 = 3$ and $c_2 = 4$.

Practice 8. What is the smallest value a Riemann sum for $f(x) = \frac{1}{x}$ can have using the partition $\{1,4,5\}$? (You will need to choose values for c_1 and c_2.) What is the largest value a Riemann sum can have for this function and partition?

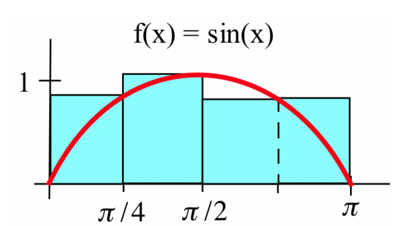
The table below shows the output of a computer program that calculated Riemann sums for the function $f(x) = \frac{1}{x}$ with various numbers of subintervals (denoted n) and different ways of choosing the points c_k in each subinterval.

n	mesh	c_k = left edge = x_{k-1}	c_k = "random" point	c_k = right edge = x_k
4	1.0	2.083333	1.473523	1.283333
8	0.5	1.828968	1.633204	1.428968
16	0.25	1.714406	1.577806	1.514406
40	0.10	1.650237	1.606364	1.570237
400	0.01	1.613446	1.609221	1.605446
4000	0.001	1.609838	1.609436	1.609038

When the mesh of the partition is small (and the number of subintervals, n, is large), it appears that all of the ways of choosing the c_k locations result in approximately the same value for the Riemann sum. For this decreasing function, using the left endpoint of the subinterval always resulted in a sum that was larger than the area approximated by the sum. Choosing the right endpoint resulted in a value smaller than that area. Why?

As the mesh gets smaller, all of the Riemann Sums seem to be approaching the same value, approximately 1.609. (As we shall soon see, these values are all approaching $\ln(5) \approx 1.609437912$.)

Example 7. Find the Riemann sum for the function $f(x) = \sin(x)$ on the interval $[0, \pi]$ using the partition $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\right\}$ and the chosen points $c_1 = \frac{\pi}{4}$, $c_2 = \frac{\pi}{2}$ and $c_3 = \frac{3\pi}{4}$.



Solution. The three subintervals (see margin) are $\left[0, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$ so $\Delta x_1 = \frac{\pi}{4}$, $\Delta x_2 = \frac{\pi}{4}$ and $\Delta x_3 = \frac{\pi}{2}$. The Riemann sum for this partition is:

$$\sum_{k=1}^{3} f(c_k) \cdot \Delta x_k = \sin\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{4} + \sin\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{4} + \sin\left(\frac{3\pi}{4}\right) \cdot \frac{\pi}{2}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} + 1 \cdot \frac{\pi}{4} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{2} = \frac{(2+3\sqrt{2})\pi}{8}$$

or approximately 2.45148. ◀

Practice 9. Find the Riemann sum for the function and partition in the previous Example, but this time choose $c_1 = 0$, $c_2 = \frac{\pi}{2}$ and $c_3 = \frac{\pi}{2}$.

Two Special Riemann Sums: Lower and Upper Sums

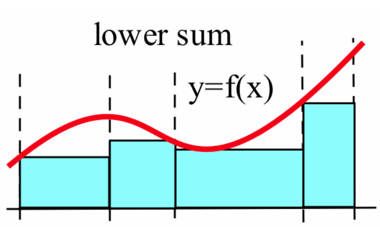
Two particular Riemann sums are of special interest because they represent the extreme possibilities for a given partition.

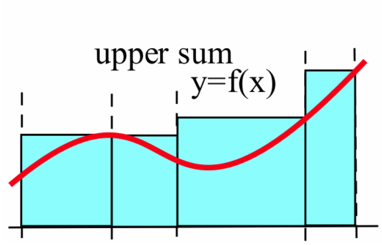
We need f to be continuous in order to assure that it attains its minimum and maximum values on any closed subinterval of the partition. If f is bounded — but not necessarily continuous — we can generalize this definition by replacing $f(m_k)$ with the **greatest lower bound** of all $f(x)$ on the interval and $f(M_k)$ with the **least upper bound** of all $f(x)$ on the interval.

Definition:
If f is a positive, continuous function on $[a,b]$ and $\mathcal{P}$ is a partition of $[a,b]$, let m_k be the x-value in the k-th subinterval so that $f(m_k)$ is the minimum value of f on that interval, and let M_k be the x-value in the k-th subinterval so that $f(M_k)$ is the maximum value of f on that subinterval. Then:

$$\text{LS}_{\mathcal{P}} = \sum_{k=1}^{n} f(m_k) \cdot \Delta x_k \text{ is the } \textbf{lower sum} \text{ of } f \text{ for } \mathcal{P}$$

$$\text{US}_{\mathcal{P}} = \sum_{k=1}^{n} f(M_k) \cdot \Delta x_k \text{ is the } \textbf{upper sum} \text{ of } f \text{ for } \mathcal{P}$$





Geometrically, a lower sum arises from building rectangles under the graph of f (see first margin figure) and *every* lower sum is less than or equal to the exact area A of the region bounded by the graph of f and the x-axis on the interval $[a,b]$: $\text{LS}_{\mathcal{P}} \le A$ for every partition $\mathcal{P}$.

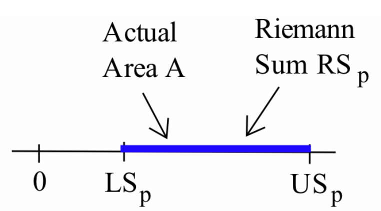
Likewise, an upper sum arises from building rectangles over the graph of f (see second margin figure) and *every* upper sum is greater than or equal to the exact area A of the region bounded by the graph of f and the x-axis on the interval $[a,b]$: $\text{US}_{\mathcal{P}} \ge A$ for every partition $\mathcal{P}$.

Together, the lower and upper sums provide bounds on the size of the exact area: $\text{LS}_{\mathcal{P}} \le A \le \text{US}_{\mathcal{P}}$.

For any c_k value in the k-th subinterval, $f(m_k) \le f(c_k) \le f(M_k)$, so, for *any* choice of the c_k values, the Riemann sum $\text{RS}_{\mathcal{P}} = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ satisfies the inequality:

$$\sum_{k=1}^{n} f(m_k) \cdot \Delta x_k \quad \le \quad \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \quad \le \quad \sum_{k=1}^{n} f(M_k) \cdot \Delta x_k$$

or, equivalently, $\text{LS}_{\mathcal{P}} \le \text{RS}_{\mathcal{P}} \le \text{US}_{\mathcal{P}}$. The lower and upper sums provide bounds on the size of *all* Riemann sums for a given partition.



The exact area A and every Riemann sum $\text{RS}_{\mathcal{P}}$ for partition $\mathcal{P}$ and any choice of points $\{c_k\}$ both lie between the lower sum and the upper sum for $\mathcal{P}$ (see margin). Therefore, if the lower and upper sums are close together, then the area and *any* Riemann sum for $\mathcal{P}$ (regardless of how you choose the points c_k) must also be close together. If we know that the upper and lower sums for a partition $\mathcal{P}$ are within 0.001 units of each other, then we can be sure that every Riemann sum for partition $\mathcal{P}$ is within 0.001 units of the exact area A.

Unfortunately, finding minimums and maximums for each subinterval of a partition can be a time-consuming (and tedious) task, so it is usually not practical to determine lower and upper sums for "wiggly" functions. If f is monotonic, however, then it is *easy* to find the values for m_k and M_k, and sometimes we can even explicitly calculate the limits of the lower and upper sums.

For a **monotonic**, bounded function we can guarantee that a Riemann sum is within a certain distance of the exact value of the area it is approximating.

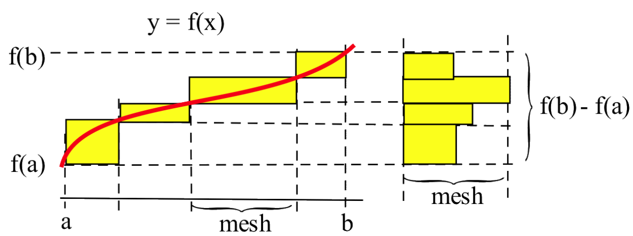
Recall from Section 3.3 that "monotonic" means "always increasing or always decreasing" on the interval in question.

Theorem:
If f is a positive, monotonic, bounded function on $[a, b]$
then for any partition $\mathcal{P}$ and any Riemann sum for f using $\mathcal{P}$,

$$|\text{RS}_{\mathcal{P}} - A| \leq \text{US}_{\mathcal{P}} - \text{LS}_{\mathcal{P}} \leq |f(b) - f(a)| \cdot (\text{mesh of } \mathcal{P})$$

In words, this string of inequalities says that the distance between any Riemann sum and the area being approximated is no bigger than the difference between the upper and lower Riemann sums for the same partition, which in turn is no bigger than the distance between the values of the function at the endpoints of the interval times the mesh of the partition.

Proof. The Riemann sum and the exact area are both between the upper and lower sums, so the distance between the Riemann sum and the exact area is no bigger than the distance between the upper and lower sums. If f is monotonically increasing, we can slide the areas representing the difference of the upper and lower sums into a rectangle:



See Problem 56 for the monotonically decreasing case.

whose height equals $f(b) - f(a)$ and whose base equals the mesh of $\mathcal{P}$. So the total difference of the upper and lower sums is smaller than the area of that rectangle, $[f(b) - f(a)] \cdot (\text{mesh of } \mathcal{P})$. □

4.1 Problems

In Problems 1–6 , rewrite the sigma notation as a summation and perform the indicated addition.

3. $\sum_{n=1}^{3} (1+n)^2$

4. $\sum_{k=0}^{5} \sin(\pi k)$

1. $\sum_{k=2}^{4} k^2$

2. $\sum_{j=1}^{5} (1+j)$

5. $\sum_{j=0}^{5} \cos(\pi j)$

6. $\sum_{k=1}^{3} \frac{1}{k}$

In Problems 7–12, rewrite each summation using the sigma notation. Do not evaluate the sums.

7. $3+4+5+\cdots+93+94$

8. $4+6+8+\cdots+24$

9. $9+16+25+36+\cdots+144$

10. $\frac{3}{4}+\frac{3}{9}+\frac{3}{16}+\cdots+\frac{3}{100}$

11. $1\cdot 2^1+2\cdot 2^2+3\cdot 2^3+\cdots+7\cdot 2^7$

12. $3+6+9+\cdots+30$

In Problems 13–15, use this table:

k	a_k	b_k
1	1	2
2	2	2
3	3	2

to verify the equality for these values of a_k and b_k.

13. $\sum_{k=1}^{3}(a_k+b_k)=\sum_{k=1}^{3}a_k+\sum_{k=1}^{3}b_k$

14. $\sum_{k=1}^{3}(a_k-b_k)=\sum_{k=1}^{3}a_k-\sum_{k=1}^{3}b_k$

15. $\sum_{k=1}^{3}5a_k=5\cdot\sum_{k=1}^{3}a_k$

For Problems 16–18, use the values of a_k and b_k in the table above to verify the inequality.

16. $\sum_{k=1}^{3}a_k\cdot b_k\neq\left(\sum_{k=1}^{3}a_k\right)\left(\sum_{k=1}^{3}b_k\right)$

17. $\sum_{k=1}^{3}a_k^2\neq\left(\sum_{k=1}^{3}a_k\right)^2$

18. $\sum_{k=1}^{3}\frac{a_k}{b_k}\neq\frac{\sum_{k=1}^{3}a_k}{\sum_{k=1}^{3}b_k}$

For 19–30, $f(x)=x^2$, $g(x)=3x$ and $h(x)=\frac{2}{x}$. Evaluate each sum.

19. $\sum_{k=0}^{3}f(k)$

20. $\sum_{k=0}^{3}f(2k)$

21. $\sum_{j=0}^{3}2\cdot f(j)$

22. $\sum_{i=0}^{3}f(1+i)$

23. $\sum_{m=1}^{3}g(m)$

24. $\sum_{k=1}^{3}g(f(k))$

25. $\sum_{j=1}^{3}g^2(j)$

26. $\sum_{k=1}^{3}k\cdot g(k)$

27. $\sum_{k=2}^{4}h(k)$

28. $\sum_{i=1}^{4}h(3i)$

29. $\sum_{n=1}^{3}f(n)\cdot h(n)$

30. $\sum_{k=1}^{7}g(k)\cdot h(k)$

In 31–36, write out each summation and simplify the result. These are examples of "telescoping sums."

31. $\sum_{k=1}^{7}\left[k^2-(k-1)^2\right]$

32. $\sum_{k=1}^{6}\left[k^3-(k-1)^3\right]$

33. $\sum_{k=1}^{5}\left[\frac{1}{k}-\frac{1}{k+1}\right]$

34. $\sum_{k=0}^{4}\left[(k+1)^3-k^3\right]$

35. $\sum_{k=0}^{8}\left[\sqrt{k+1}-\sqrt{k}\right]$

36. $\sum_{k=1}^{5}[x_k-x_{k-1}]$

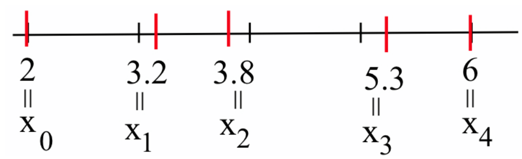
In 37–43, (a) list the subintervals determined by the partition $\mathcal{P}$, (b) find the values of Δx_k, (c) find the mesh of $\mathcal{P}$ and (d) calculate $\sum_{k=1}^{n}\Delta x_k$.

37. $\mathcal{P}=\{2,3,4.5,6,7\}$

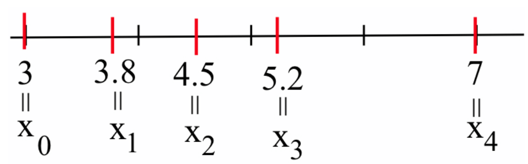
38. $\mathcal{P}=\{3,3.6,4,4.2,5,5.5,6\}$

39. $\mathcal{P}=\{-3,-1,0,1.5,2\}$

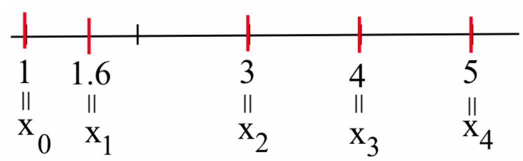
40. $\mathcal{P}$ as shown below:



41. $\mathcal{P}$ as shown below:



42. $\mathcal{P}$ as shown below:



43. For $\Delta x_k = x_k - x_{k-1}$, verify that:

$$\sum_{k=1}^{n} \Delta x_k = \text{length of the interval } [a,b]$$

For 44–48, sketch a graph of f, draw vertical lines at each point of the partition, evaluate each $f(c_k)$ and sketch the corresponding rectangle, and calculate and add up the areas of those rectangles.

44. $f(x) = x+1$, $\mathcal{P} = \{1,2,3,4\}$
   (a) $c_1 = 1$, $c_2 = 3$, $c_3 = 3$
   (b) $c_1 = 2$, $c_2 = 2$, $c_3 = 4$

45. $f(x) = 4 - x^2$, $\mathcal{P} = \{0,1,1.5,2\}$
   (a) $c_1 = 0$, $c_2 = 1$, $c_3 = 2$
   (b) $c_1 = 1$, $c_2 = 1.5$, $c_3 = 1.5$

46. $f(x) = \sqrt{x}$, $\mathcal{P} = \{0,2,5,10\}$
   (a) $c_1 = 1$, $c_2 = 4$, $c_3 = 9$
   (b) $c_1 = 0$, $c_2 = 3$, $c_3 = 7$

47. $f(x) = \sin(x)$, $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\}$
   (a) $c_1 = 0$, $c_2 = \dfrac{\pi}{4}$, $c_3 = \dfrac{\pi}{2}$
   (b) $c_1 = \dfrac{\pi}{4}$, $c_2 = \dfrac{\pi}{2}$, $c_3 = \pi$

48. $f(x) = 2^x$, $\mathcal{P} = \{0,1,3\}$
   (a) $c_1 = 0$, $c_2 = 2$
   (b) $c_1 = 1$, $c_2 = 3$

For 49–52, sketch the function and find the smallest possible value and the largest possible value for a Riemann sum for the given function and partition.

49. $f(x) = 1 + x^2$
   (a) $\mathcal{P} = \{1,2,4,5\}$
   (b) $\mathcal{P} = \{1,2,3,4,5\}$
   (c) $\mathcal{P} = \{1,1.5,2,3,4,5\}$

50. $f(x) = 7 - 2x$
   (a) $\mathcal{P} = \{0,2,3\}$
   (b) $\mathcal{P} = \{0,1,2,3\}$
   (c) $\mathcal{P} = \{0,.5,1,1.5,2,3\}$

51. $f(x) = \sin(x)$
   (a) $\mathcal{P} = \{0, \frac{\pi}{2}, \pi\}$
   (b) $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\}$
   (c) $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi\}$

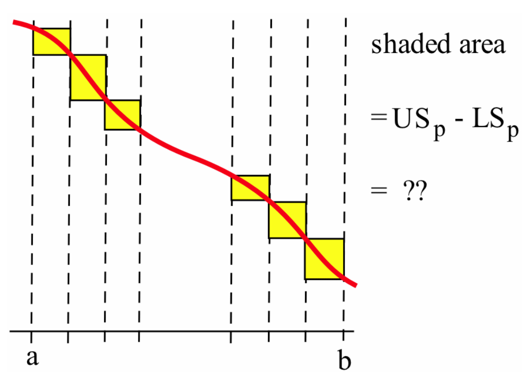
52. $f(x) = x^2 - 2x + 3$
   (a) $\mathcal{P} = \{0,2,3\}$
   (b) $\mathcal{P} = \{0,1,2,3\}$
   (c) $\mathcal{P} = \{0,0.5,1,2,2.5,3\}$

53. Suppose $LS_{\mathcal{P}} = 7.362$ and $US_{\mathcal{P}} = 7.402$ for a positive function f and a partition $\mathcal{P}$ of $[1,5]$.
   (a) You can be certain that every Riemann sum for the partition $\mathcal{P}$ is within what distance of the exact value of the area between the graph of f and the x-axis on the interval $[1,5]$?
   (b) What if $LS_{\mathcal{P}} = 7.372$ and $US_{\mathcal{P}} = 7.390$?

54. Suppose you divide the interval $[1,4]$ into 100 equally wide subintervals and calculate a Riemann sum for $f(x) = 1 + x^2$ by randomly selecting a point c_k in each subinterval.
   (a) You can be certain that the value of the Riemann sum is within what distance of the exact value of the area between the graph of f and the x-axis on interval $[1,4]$?
   (b) What if you use 200 equally wide subintervals?

55. If you divide $[2,4]$ into 50 equally wide subintervals and calculate a Riemann sum for $f(x) = 1 + x^3$ by randomly selecting a point c_k in each subinterval, then you can be certain that the Riemann sum is within what distance of the exact value of the area between f and the x-axis on the interval $[2,4]$?

56. If f is monotonic decreasing on $[a,b]$ and you divide $[a,b]$ into n equally wide subintervals:



then you can be certain that the Riemann sum is within what distance of the exact value of the area between f and the x-axis on the interval $[a,b]$?

Summing Powers of Consecutive Integers

The formulas below are included here for your reference. They will not be used in the following sections, except for a handful of exercises in Section 4.2.

Formulas for some commonly encountered summations can be useful for explicitly evaluating certain special Riemann sums.

The summation formula for the first n positive integers is relatively well known, has several easy but clever proofs, and even has an interesting story behind it.

$$1+2+3+\cdots+(n-1)+n=\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

Proof. Let $S=1+2+3+\cdots+(n-2)+(n-1)+n$, which we can also write as $S=n+(n-1)+(n-2)+\cdots+3+2+1$. Adding these two representations of S together:

$$\begin{array}{rccccccccccccc} S= & 1 & + & 2 & + & 3 & + & \cdots & + & (n-2) & + & (n-1) & + & n \\ +\,S= & n & + & (n-1) & + & (n-2) & + & \cdots & + & 3 & + & 2 & + & 1 \\ \hline 2S= & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) & + & (n+1) \end{array}$$

So $2S=n\cdot(n+1)\Rightarrow S=\dfrac{n(n+1)}{2}$, the desired formula. □

Karl Friedrich Gauss (1777–1855), a German mathematician sometimes called the "prince of mathematics."

Gauss supposedly discovered this formula for himself at the age of five when his teacher, planning to keep the class busy for a while, asked the students to add up the integers from 1 to 100. Gauss thought a few minutes, wrote his answer on his slate, and turned it in, then sat smugly while his classmates struggled with the problem.

57. Find the sum of the first 100 positive integers in two ways: (a) using Gauss' formula, and (b) using Gauss' method (from the proof).

58. Find the sum of the first 10 odd integers. (Each odd integer can be written in the form $2k-1$ for $k=1,2,3,\ldots$.)

59. Find the sum of the integers from 10 to 20.

Formulas for other integer powers of the first n integers are also known:

$$\sum_{k=1}^{n} k = \frac{1}{2}n^2+\frac{1}{2}n=\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n=\frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2=\left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{k=1}^{n} k^4 = \frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

In Problems 60–62, use the properties of summation and the formulas for powers given above to evaluate each sum.

60. $\sum_{k=1}^{10} \left(3+2k+k^2\right)$ 61. $\sum_{k=1}^{10} k \cdot \left(k^2+1\right)$ 62. $\sum_{k=1}^{10} k^2 \cdot (k-3)$

4.1 Practice Answers

1. (a) $\sum_{k=1}^{5} k^3 = 1+8+27+64+125$

   (b) $\sum_{j=2}^{7} (-1)^j \cdot \frac{1}{j} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7}$

   (c) $\sum_{m=0}^{4} (2m+1) = 1+3+5+7+9$

2. (a) $\sum_{k=2}^{5} g(k) = g(2)+g(3)+g(4)+g(5) = 1+(-2)+3+5 = 7$

   (b) $\sum_{j=1}^{4} h(j) = h(1)+h(2)+h(3)+h(4) = 3+3+3+3 = 12$

   (c) $\sum_{k=3}^{5} (g(k)+f(k-1)) = (g(3)+f(2))+(g(4)+f(3))+(g(5)+f(4))$
   $= (-2+3)+(3+1)+(5+0) = 10$

3. $\sum_{k=2}^{4} g(k) = g(2)+g(3)+g(4) = \frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{13}{12}$
   $\sum_{k=1}^{3} g(k+1) = g(2)+g(3)+g(4) = \frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{13}{12}$

4. Rectangular areas $= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{25}{12} = \sum_{j=1}^{4} \frac{1}{j}$

5. $f(x_0) \cdot (x_1 - x_0) + f(x_1) \cdot (x_2 - x_1) + f(x_2) \cdot (x_3 - x_2) = \sum_{j=1}^{3} f(x_{j-1}) \cdot (x_j - x_{j-1})$
   or $\sum_{k=0}^{2} f(x_k) \cdot (x_{k+1} - x_k)$

6. Interval is $[3,8]$; six subintervals; mesh $= 1.2$; $x_2 = 4.8$; $\Delta x_2 = x_2 - x_1 = 4.8 - 3.8 = 1$.

7. $\text{RS} = (3)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{4}\right) = 1.25$

8. smallest $\text{RS} = (3)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{5}\right) = 0.95$
   largest $\text{RS} = (3)(1) + (1)\left(\frac{1}{4}\right) = 3.25$

9. $\text{RS} = (0)\left(\frac{\pi}{4}\right) + (1)\left(\frac{\pi}{4}\right) + (1)\left(\frac{\pi}{2}\right) \approx 2.356$

4.2 *The Definite Integral*

Each particular Riemann sum depends on several things: the function f, the interval $[a,b]$, the partition $\mathcal{P}$ of that interval, and the chosen values c_k from each subinterval of that partition. Fortunately — for most of the functions needed for applications — as the approximating rectangles get thinner (and as the meshes of the partitions $\mathcal{P}$ approach 0 and the number of subintervals n in those partitions approaches ∞) the values of the Riemann sums approach the same number, independent of the particular partitions P and the chosen points c_k in the subintervals of those partitions.

This limit of the Riemann sums will become the next big topic in calculus: the **definite integral**. Integrals arise throughout the rest of this book and in applications in almost every field that uses mathematics.

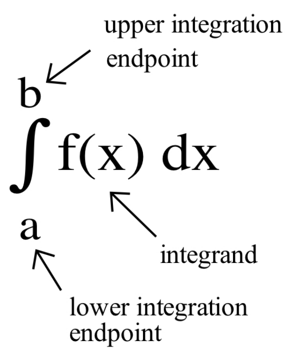
Here we use the notation $\|\mathcal{P}\|$ to mean "the mesh of $\mathcal{P}$."

Definition of the Definite Integral:

If $\displaystyle\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k\right)$ equals a finite number I, where each $\mathcal{P}$ is a partition of the interval $[a,b]$, then we say f is **integrable** on the interval $[a,b]$ and call the number I the **definite integral** of f on $[a,b]$ and write it as $\displaystyle\int_a^b f(x)\,dx$.

The dx is a differential (see Section 3.6), the limit of the discrete quantity Δx in the Riemann sum.

We read the symbol $\displaystyle\int_a^b f(x)\,dx$ as "the integral from a to b of 'eff' of x 'dee' x" or "the integral from a to b of $f(x)$ with respect to x." Furthermore, we call $f(x)$ the **integrand**, a the **lower endpoint of integration** and b the **upper endpoint** of integration. (We will sometimes also call a and b the **upper and lower limits** of integration.)



Example 1. Describe the area between the graph of $f(x) = \dfrac{1}{x}$, the x-axis, and the vertical lines at $x = 1$ and $x = 5$ as a limit of Riemann sums and as a definite integral.

Solution. Here $f(x) = \dfrac{1}{x}$, $a = 1$ and $b = 5$, so:

$$\text{area} = \lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}\frac{1}{c_k}\cdot\Delta x_k\right) = \int_1^5 \frac{1}{x}\,dx$$

which, according to estimations made in Section 4.1, is approximately equal to 1.609. ◀

You may have noticed that we did not precisely define what $\displaystyle\lim_{\|\mathcal{P}\|\to 0}$ means or how to compute this limit. Providing a definition turns out to be more complicated than the limits we have encountered so far, and in practice we will rarely need to compute such a limit, so a formal definition is left to more advanced textbooks.

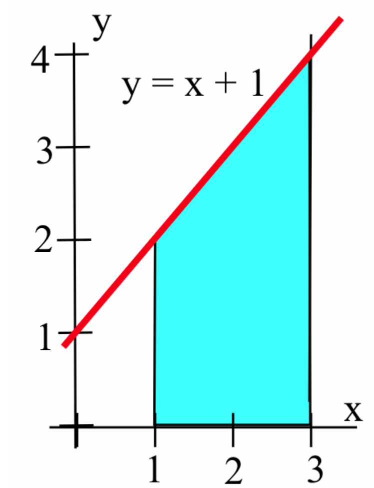
Practice 1. Describe the area between the graph of $f(x) = \sin(x)$, the x-axis, and the vertical lines at $x = 0$ and $x = \pi$ as a limit of Riemann sums and as a definite integral.

Example 2. Using the concept of area, determine the values of:

(a) $\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}(1+c_k)\cdot\Delta x_k\right)$ on the interval $[1,3]$

(b) $\int_0^4 (5-x)\,dx$

(c) $\int_{-1}^{1}\sqrt{1-x^2}\,dx$



Solution. (a) The limit of the Riemann sums represents the area between the graph of $f(x)=1+x$, the x-axis, and the vertical lines at $x=1$ and $x=3$ (see margin); this area equals 6 square units.

(b) The definite integral represents the area between $f(x)=5-x$, the x-axis and the vertical lines at $x=0$ and $x=4$, which is a trapezoid with area 12 square units.

(c) The definite integral represents the area of the upper half of the circle $x^2+y^2=1$, which has radius 1 and center at $(0,0)$. The area of this semicircle is $\frac{1}{2}\cdot\pi r^2=\frac{1}{2}\cdot\pi\cdot 1^2=\frac{\pi}{2}$. ◀

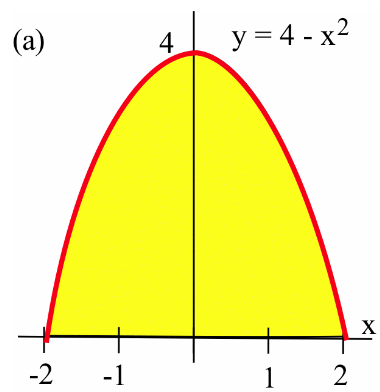
Practice 2. Using the concept of area, determine the values of:

(a) $\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}(2c_k)\cdot\Delta x_k\right)$ on the interval $[1,3]$ (b) $\int_3^8 4\,dx$

Example 3. Represent each limit of Riemann sums as a definite integral.

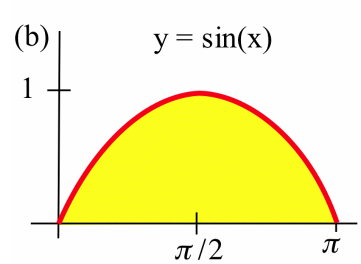
(a) $\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}(3+c_k)\,\Delta x_k\right)$ on $[1,4]$ (b) $\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}\sqrt{c_k}\,\Delta x_k\right)$ on $[0,9]$

Solution. (a) $\int_1^4 (3+x)\,dx$ (b) $\int_0^9 \sqrt{x}\,dx$ ◀



Example 4. Represent each shaded area in the margin figure as a definite integral. (Do not attempt to evaluate the definite integral, just translate the picture into symbols.)

Solution. (a) $\int_{-2}^{2}\left(4-x^2\right)\,dx$ (b) $\int_0^{\pi}\sin(x)\,dx$ ◀

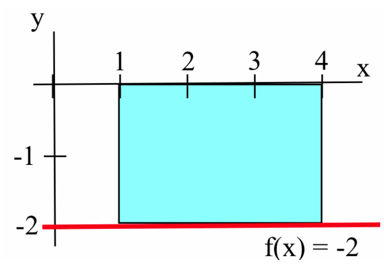


The value of a definite integral $\int_a^b f(x)\,dx$ depends only on the function f being integrated and on the interval $[a,b]$. Replacing the variable x that appears in $\int_a^b f(x)\,dx$, sometimes called a "dummy variable," does not change the value of the integral. The following definite integrals each represent the integral of the function f on the interval $[a,b]$, and they are all equal:

$$\int_a^b f(x)\,dx \quad = \quad \int_a^b f(t)\,dt \quad = \quad \int_a^b f(u)\,du \quad = \quad \int_a^b f(w)\,dw$$

Definite Integrals of Negative Functions

A definite integral is a limit of Riemann sums, and you can form Riemann sums using any integrand function f, positive or negative (or both), continuous or discontinuous. The definite integral of an integrable function will still have a geometric meaning even if the function is sometimes (or always) negative, and definite integrals of negative functions also have meaningful interpretations in applications.



Example 5. Find the definite integral of $f(x) = -2$ on $[1, 4]$.

Solution. Writing a Riemann sum for $f(x) = -2$ on the interval $[1, 4]$:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} (-2) \cdot \Delta x_k = -2 \cdot \sum_{k=1}^{n} \Delta x_k = -2(3) = -6$$

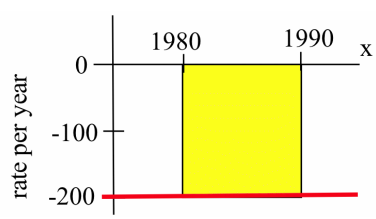
for every partition $\mathcal{P}$ and every choice of values for c_k so:

$$\int_1^4 -2\,dx = \lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} (-2) \cdot \Delta x_k \right) = \lim_{\|\mathcal{P}\| \to 0} -6 = -6$$

The **area** of the region in in the margin figure is 6 units, but because the region is below the x-axis, the value of the **integral** is -6. ◀

If the graph of $f(x)$ is below the x-axis for $a \le x \le b$ (f is negative) then $\int_a^b f(x)\,dx$ is -1 times the area of the region below the x-axis and above the graph of $f(x)$ between $x = a$ and $x = b$.

If $f(t)$ represents the rate of population change (people per year) for a town, then negative values of f for a given time interval would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would represent the change in the population—a decrease—during that time interval.



Example 6. In 1980 there were 12,000 ducks nesting around a lake. The **rate** of population change is shown in the margin. Write a definite integral to represent the **total change** in the duck population between 1980 and 1990, then estimate the population in 1990.

Solution. The total change in population is given by $\int_{1980}^{1990} f(t)\,dt$ and this definite integral is equal to -1 times the area of the rectangle in the margin figure:

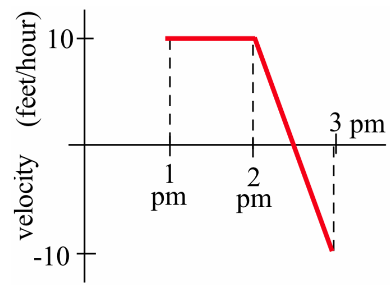
$$-200\,\frac{\text{ducks}}{\text{year}} \cdot 10\,\text{years} = -2000\,\text{ducks}$$

so:

$$\begin{aligned}[\text{1990 population}] &= [\text{1980 population}] + [\text{change from 1980 to 1990}] \\ &= 12000 + (-2000) = 10000\end{aligned}$$

Approximately 10,000 ducks were nesting around the lake in 1990. ◀

If $f(t)$ represents the velocity of a car (in miles per hour) moving in the positive direction along a straight line at time t, then negative values of f indicate that the car was travelling in the negative direction (that is, backwards). The definite integral of f (the integral will be a negative number) represents the change in position of the car during the time interval: how far the car travelled in the negative direction.

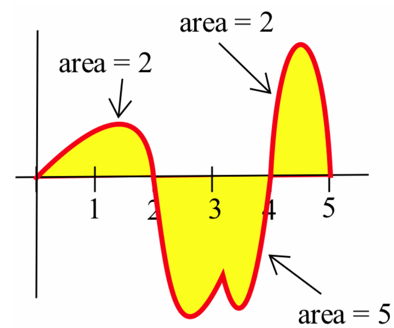


Practice 3. A bug starts at the location $x = 12$ on the x-axis at 1:00 p.m. and walks along the axis with the velocity shown in the margin figure. How far does the bug travel between 1:00 p.m. and 3:00 p.m.? Where is the bug at 3:00 p.m.?

Frequently an integrand function will be positive some of the time and negative some of the time. If f represents a rate of population increase, then the integral of the positive parts of f will give the increase in population and the integral of the negative parts of f will give the decrease in population. Altogether, the integral of f over the entire time interval will give the **total (net) change** in the population.

Geometrically, we can now interpret a definite integral as a difference of areas of the region(s) between the graph of f and the horizontal axis:

$$\int_a^b f(x)\,dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}]$$



Example 7. Use the margin figure to calculate $\int_0^2 f(x)\,dx$, $\int_2^4 f(x)\,dx$, $\int_4^5 f(x)\,dx$ and $\int_0^5 f(x)\,dx$.

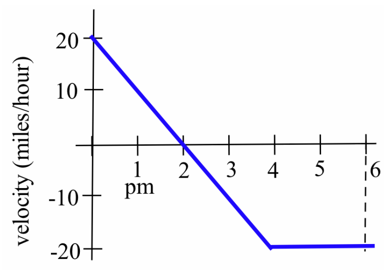
Solution. Using the areas indicated in the figure, $\int_0^2 f(x)\,dx = 2$, $\int_2^4 f(x)\,dx = -5$ and $\int_4^5 f(x)\,dx = 2$, while

$$\begin{aligned}\int_0^5 f(x)\,dx &= [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}] \\ &= [2+2] - [5] = -1\end{aligned}$$

where we added the areas of the regions above the x-axis and subtracted the area of the region below the x-axis. ◀

Practice 4. Use geometric reasoning to evaluate $\int_0^{2\pi} \sin(x)\,dx$.

If f represents a velocity, then integrals on the intervals where f is positive measure distances moved in the forward direction and integrals on the intervals where f is negative measure distances moved in the backward direction. The integral over the whole time interval gives the **total (net) change** in position: the distance moved forward minus the distance moved backward.



Practice 5. A car travels west with the velocity shown in the margin.

(a) How far does the car travel between noon and 6:00 p.m.?

(b) At 6:00 p.m., where is the car relative to its position at noon?

Units for the Definite Integral

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if x measures time in "seconds" and $f(x)$ gives a velocity with units "feet per second," then Δx has the units "seconds" and $f(x) \cdot \Delta x$ has units:

$$\left(\frac{\text{feet}}{\text{second}}\right)(\text{seconds}) = \text{feet}$$

which is a measure of distance. Because each Riemann sum $\sum f(x) \cdot \Delta x$ is a sum of "feet" and the definite integral is a limit of these Riemann sums, the definite integral has the same units, "feet."

If the units of $f(x)$ are "square feet" and the units of x are "feet," then $\int_a^b f(x)\,dx$ is a number with the units $(\text{feet}^2) \cdot (\text{feet}) = \text{feet}^3$, or cubic feet, a measure of volume. If $f(x)$ represents a force in pounds and x is a distance in feet, then $\int_a^b f(x)\,dx$ is a number with the units foot-pounds, a measure of work.

In general, the units for $\int_a^b f(x)\,dx$ are (units for $f(x)$) $\cdot$ (units for x). A quick check of the units when computing a definite integral can help avoid errors when solving an applied problem.

4.2 Problems

In Problems 1–4 , rewrite each limit of Riemann sums as a definite integral.

1. $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} (2 + 3c_k) \cdot \Delta x_k \right)$ on $[0, 4]$

2. $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} (c_k)^3 \cdot \Delta x_k \right)$ on $[0, 11]$

3. $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} \cos(5c_k) \cdot \Delta x_k \right)$ on $[2, 5]$

4. $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} \sqrt{c_k} \cdot \Delta x_k \right)$ on $[1, 4]$

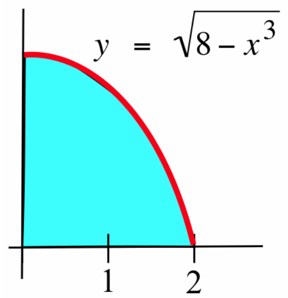
In Problems 5–10, represent the area of each bounded region as a definite integral. (Do not attempt to evaluate the integral, just translate the area into an integral.)

5. The region bounded by $y = x^3$, the x-axis, and the lines $x = 1$ and $x = 5$.

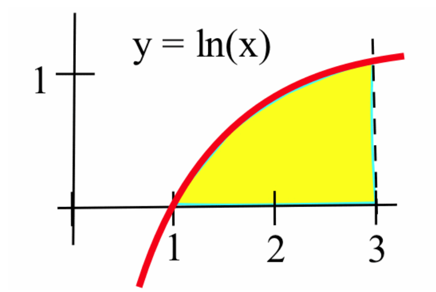
6. The region bounded by $y = \sqrt{x}$, the x-axis and the line $x = 9$.

7. The region bounded by $y = x \cdot \sin(x)$, the x-axis, and the lines $x = \frac{1}{2}$ and $x = 2$.

8. The shaded region shown below:



9. The shaded region shown below:



10. The shaded region shown above for $2 \le x \le 3$.

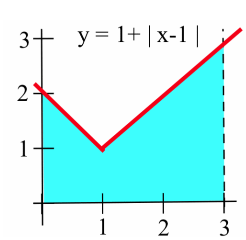
In Problems 11–15, represent the area of each bounded region as a definite integral, then use geometry to determine the value of that definite integral.

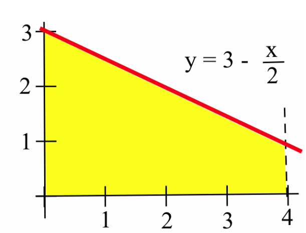
11. The region bounded by $y = 2x$, the x-axis, and the lines $x = 1$ and $x = 3$.

12. The region bounded by $y = 4 - 2x$, the x-axis and the y-axis.

13. The region bounded by $y = |x|$, the x-axis and the line $x = -1$.

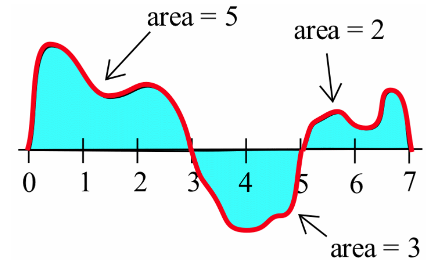
14. The shaded region shown below left.





15. The shaded region shown above right.

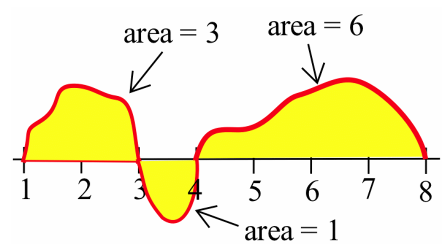
16. Evaluate each integral using the figure below showing the graph of f and various areas.



(a) $\int_0^5 f(x)\,dx$ (b) $\int_3^5 f(x)\,dx$ (c) $\int_5^7 f(x)\,dx$

(d) $\int_0^5 |f(x)|\,dx$ (e) $\int_3^7 f(x)\,dx$

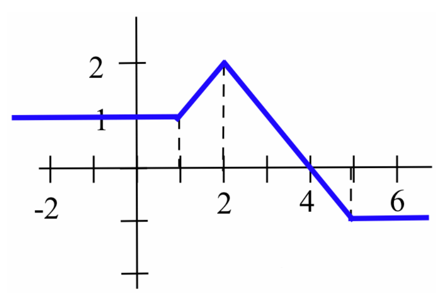
17. Evaluate each integral using the figure below showing the graph of g and various areas.



(a) $\int_1^3 g(x)\,dx$ (b) $\int_3^4 g(x)\,dx$ (c) $\int_4^8 g(x)\,dx$

(d) $\int_1^8 g(x)\,dx$ (e) $\int_3^8 |g(x)|\,dx$

18. Evaluate each integral using the figure below showing the graph of h.

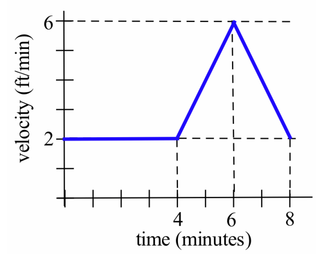


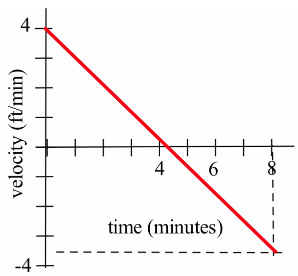
(a) $\int_{-1}^1 h(x)\,dx$ (b) $\int_4^6 h(x)\,dx$ (c) $\int_{-2}^6 h(x)\,dx$

(d) $\int_{-2}^4 h(x)\,dx$ (e) $\int_{-2}^4 |h(x)|\,dx$

For Problems 19–20, the figure shows your velocity (in feet per minute) along a straight path. (a) Sketch a graph of your location. (b) How many feet did you walk in 8 minutes? (c) Where, relative to your starting location, are you after 8 minutes?

19. See figure below left.





20. See figure above right.

Problems 21–27 give the units for x and $f(x)$. Specify the units of the definite integral $\int_a^b f(x)\,dx$.

21. x is time in "seconds"; $f(x)$ is velocity in "meters per second"
22. x is time in "hours"; $f(x)$ is a flow rate in "gallons per hour"
23. x a position in "feet"; $f(x)$ area in "square feet"
24. x is a time in "days"; $f(x)$ is a temperature in "degrees Celsius"
25. x a height in "meters"; $f(x)$ force in "grams"
26. x is a position in "inches"; $f(x)$ is a density in "pounds per inch"
27. x is a time in "seconds"; $f(x)$ is an acceleration in "feet per second per second" $\left(\frac{\text{ft}}{\text{sec}^2}\right)$

The remaining problems use the summation formulas given at the end of Section 4.1, as demonstrated in the following Example.

Example 8. For $f(x) = x^2$, divide the interval $[0, 2]$ into n equally wide subintervals, evaluate the lower sum, and compute the limit of that lower sum as $n \to \infty$.

Solution. The width of the interval is $b - a = 2 - 0 = 2$ so each of the n subintervals should have width $\Delta x = \dfrac{b-a}{n} = \dfrac{2}{n}$. The endpoints of the k-th interval in the partition are therefore $(k-1) \cdot \frac{2}{n}$ and $k \cdot \frac{2}{n}$ for $k = 1, 2, \ldots, n$.

Because $f(x) = x^2$ is increasing on $[0, 2]$ the minimum value of the function on each subinterval occurs at the left endpoint of the subinterval, hence we need to choose $c_k = (k-1) \cdot \frac{2}{n}$. So:

$$\begin{aligned}
\text{LS} &= \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} \left((k-1) \cdot \frac{2}{n} \right)^2 \cdot \frac{2}{n} = \frac{8}{n^3} \cdot \sum_{k=1}^{n} (k-1)^2 \\
&= \frac{8}{n^3} \cdot \sum_{k=1}^{n} \left(k^2 - 2k + 1 \right) = \frac{8}{n^3} \left[\sum_{k=1}^{n} k^2 \; - \; 2 \sum_{k=1}^{n} k \; + \; \sum_{k=1}^{n} 1 \right] \\
&= \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - 2 \cdot \frac{n(n+1)}{2} + n \right] = \frac{8}{n^3} \left[\frac{2n^3 - 3n^2 + n}{6} \right] \\
&= \frac{8}{6} \left[2 - \frac{3}{n} + \frac{1}{n^2} \right]
\end{aligned}$$

As $n \to \infty$, LS $\to \dfrac{8}{6}(2) = \dfrac{8}{3}$ so we can be certain that $\displaystyle\int_0^2 x^2\,dx \geq \frac{8}{3}$. ◀

Practice 6. Redo Example 6 but find the upper Riemann sum for n equally wide partition intervals and show that the limit of these upper sums, as $n \to \infty$, is $\frac{8}{3}$.

From the previous Example and Practice problem, we know that

$$\frac{8}{3} \leq \int_0^2 x^2\,dx \leq \frac{8}{3}$$

so we can conclude that $\int_0^2 x^2 = \frac{8}{3}$. We will discover a much easier method for evaluating this integral in Section 4.4.

28. For $f(x) = 3 + x$, partition the interval $[0, 2]$ into n equally wide subintervals of length $\Delta x = \frac{2}{n}$.

    (a) Compute the lower sum for this function and partition, and calculate the limit of that lower sum as $n \to \infty$.

    (b) Compute the upper sum for this function and partition and find the limit of the upper sum as $n \to \infty$.

29. For $f(x) = x^3$, partition the interval $[0, 2]$ into n equally wide subintervals of length $\Delta x = \frac{2}{n}$.

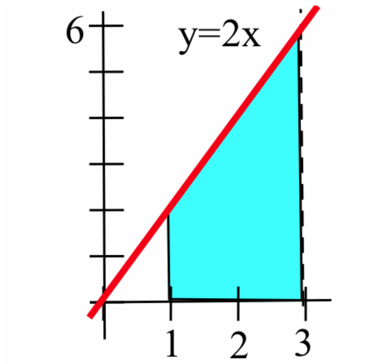
    (a) Compute the lower sum for this function and partition, and calculate the limit of that lower sum as $n \to \infty$.

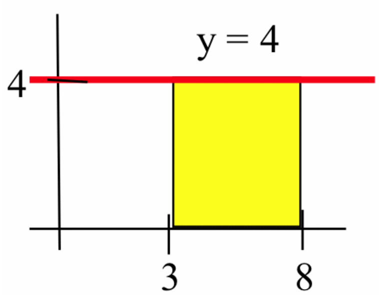
    (b) Compute the upper sum for this function and partition and find the limit of the upper sum as $n \to \infty$.

30. For $f(x) = \sqrt{x}$, partition the interval $[0, 9]$ into n subintervals by taking $x_k = \frac{9}{n^2} \cdot k^2$ for $k = 1, 2, \ldots, n$.

    (a) Choose $c_k = x_k$ for each subinterval and compute the upper sum for this function and partition, then calculate the limit of that upper sum as $n \to \infty$.

    (b) Compute the lower sum for this function and partition and find the limit of the lower sum as $n \to \infty$.





4.2 Practice Answers

1. area $= \lim_{\|\mathcal{P}\|\to 0} \left(\sum_{k=1}^{n} \sin(c_k) \cdot \Delta x_k \right) = \int_0^{\pi} \sin(x)\, dx$

2. $\lim_{\|\mathcal{P}\|\to 0} \left(\sum_{k=1}^{n} (2c_k) \cdot \Delta x_k \right) =$ area of trapezoid in margin $= 8$

   $\int_3^8 4\, dx =$ area of rectangle in margin $= 20$

3. (a) 12.5 feet forward and 2.5 feet backward $=$ 15 feet total

   (b) The bug ends up 10 feet forward of its starting position at $x = 12$, so the bug's final location is at $x = 22$.

4. Between $x = 0$ and $x = 2\pi$, the graph of $y = \sin(x)$ has the same area above the x-axis as below the x-axis so the two areas cancel and the definite integral is 0: $\int_0^{2\pi} \sin(x)\, dx = 0$.

5. (a) 20 miles west (from noon to 2:00 p.m.) plus 60 miles east (from 2:00 p.m. to 6:00 p.m.) yields a total travel distance of 80 miles. (At 4:00 p.m. the driver is back at the starting position after driving 40 miles: 20 miles west and then 20 miles east.)

   (b) The car is 40 miles east of the starting location. (East is the "negative" of west.)

6. $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $M_k = \frac{2}{n} \cdot k$ so $f(M_k) = \left(\frac{2}{n} \cdot k\right)^2 = \frac{4}{n^2} \cdot k^2$. Then:

$$\text{US} = \sum_{k=1}^{n} f(M_k) \cdot \Delta x = \sum_{k=1}^{n} \frac{4}{n^2} \cdot k^2 \cdot \frac{2}{n}$$

$$= \frac{8}{n^3} \sum_{k=1}^{n} k^2 = \frac{8}{n^3} \left[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \right] = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

   so the limit of these upper sums as $n \to \infty$ is $\frac{8}{3}$.

4.3 *Properties of the Definite Integral*

We have defined definite integrals as limits of Riemann sums, which can often be interpreted as "areas" of geometric regions. These two powerful concepts of the definite integral can help us understand integrals and use them in a variety of applications.

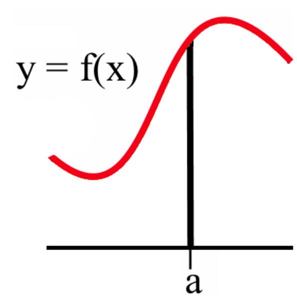
This section continues to emphasize this dual view of definite integrals and presents several properties of definite integrals. We will justify these properties using the properties of summations and the definition of a definite integral as a Riemann sum, but they also have natural interpretations as properties of areas of regions.

We will then use these properties to help understand functions that are defined by integrals, and later to help calculate the values of definite integrals.

Properties of the Definite Integral

As you read each statement about definite integrals, draw a sketch or examine the accompanying figure to interpret the property as a statement about areas.

$$\int_a^a f(x)\,dx = 0$$



This property says that the definite integral of a function over an interval consisting of a single point must be 0. Geometrically, we can see that the area under the graph of a a function above a single point should be 0 because the "width" of a point is 0. In terms of Riemann sums, we can't partition a single point, so instead we must *define* the value of any definite integral over a non-existent "interval" to be 0.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

In words, this property says that if we reverse the limits of integration, we must multiply the value of the definite integral by -1.

Geometrically, if $a < b$ then the x-values in the first integral are moving "backwards" from $x = b$ to $x = a$, so it might seem reasonable that we should get a negative answer.

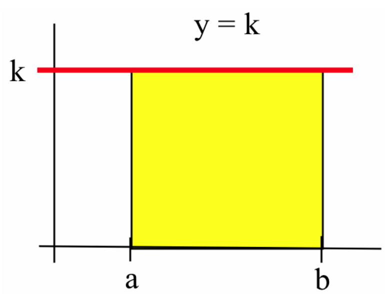
In terms of Riemann sums, if we move from right to left, each Δx_k in any partition $\mathcal{P}$ will be negative:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} f(c_k) \cdot (-|\Delta x_k|) = -1 \cdot \sum_{k=1}^{n} f(c_k) \cdot |\Delta x_k|$$

resulting in -1 times the Riemann sum we would use for $\int_a^b f(x)\,dx$.

Our definition of a Riemann sum only allows each Δx_k to be positive, however, so we can simply treat this integral property as another definition.

$$\int_a^b k\,dx = k\,(b-a) \qquad (k \text{ is any constant})$$



Here we use the fact that the sum of the lengths of the subinterval of any partition of the interval $[a,b]$ is equal to the width of $[a,b]$, which is $b-a$.

Thinking geometrically, if $k>0$ (see margin), then $\int_a^b k\,dx$ represents the area of a rectangle with base $b-a$ and height k, so:

$$\int_a^b k\,dx = (\text{height})\cdot(\text{base}) = k\cdot(b-a)$$

Alternatively, for any $\mathcal{P} = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ that partitions the interval $[a,b]$, and every choice of points c_j from the subintervals of that partition, the Riemann sum is:

$$\sum_{j=1}^{n} f\left(c_j\right)\cdot\Delta x_j = \sum_{j=1}^{n} k\cdot\Delta x_j = k\sum_{j=1}^{n}\Delta x_j = k\cdot(b-a)$$

Because every Riemann sum equals $k\cdot(b-a)$, the limit of those sums, as $\|\mathcal{P}\|\to 0$, must also be $k\cdot(b-a)$.

$$\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx \qquad (k \text{ is any constant})$$

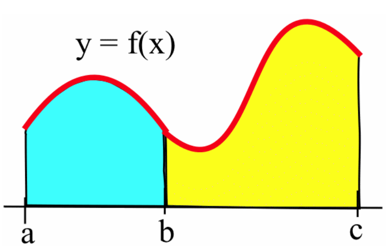
In words, this property says that multiplying an integrand by a constant k has the same result as multiplying the value of the definite integral by that constant.

Geometrically, multiplying a function by a positive constant k stretches the graph of $y=f(x)$ by a factor of k in the vertical direction, which should multiply the area of the region between that graph and the x-axis by the same factor.

Thinking in terms of Riemann sums:

$$\sum_{j=1}^{n} k\cdot f\left(c_j\right)\cdot\Delta x_j = k\cdot\sum_{j=1}^{n} f\left(c_j\right)\cdot\Delta x_j$$

so the limit of the sum on the left over all possible partitions $\mathcal{P}$, as $\|\mathcal{P}\|\to 0$, is $\int_a^b k\cdot f(x)\,dx$, while the corresponding limit of the sums on the right yields $k\cdot\int_a^b f(x)\,dx$.



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

This property is most easily understood (and believed) in terms of a picture (see margin). We can also justify this property using Riemann sums by restricting our partitions to include the point $x=b$ between $x=a$ and $x=c$ and then splitting that partition into two sub-partitions that partition $[a,b]$ and $[b,c]$, respectively.

This property remains true, however, even when $b\ge c$ or $b\le a$.

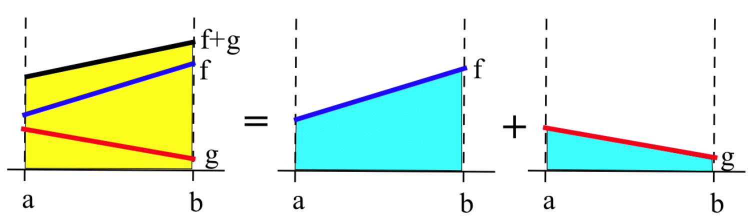
Properties of Definite Integrals of Combinations of Functions

The next two properties relate the values of integrals of sums and differences of functions to the sums and differences of integrals of the individual functions. You will find these properties very useful when computing integrals of functions that involve the sum or difference of several terms (such as a polynomial): you can integrate each term and then add or subtract the individual results to get the answer. Both properties have natural interpretations as statements about areas.

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx \; + \; \int_a^b g(x)\, dx$$

In words, this says "the integral of a sum is the sum of the integrals."

The following graph supplies a geometrical justification:



Using Riemann sums, we can write:

$$\sum_{j=1}^{n} \left[f\left(c_j\right) + g\left(c_j\right)\right] \cdot \Delta x_j = \sum_{j=1}^{n} f\left(c_j\right) \cdot \Delta x_j \; + \; \sum_{j=1}^{n} g\left(c_j\right) \cdot \Delta x_j$$

and then take the limit on each side as $\|\mathcal{P}\| \to 0$.

$$\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx \; - \; \int_a^b g(x)\, dx$$

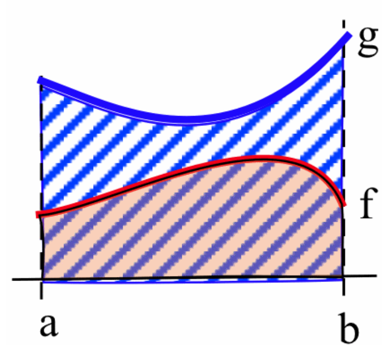
In words, this says "the integral of a difference is the difference of the integrals."

The justification for this difference property is quite similar to the justification of the sum property. (Or we can combine the sum property with the constant-multiple property, setting $k = -1$.)

Practice 1. Given that $\int_1^4 f(x)\, dx = 7$ and that $\int_1^4 g(x)\, dx = 3$, evaluate the definite integral $\int_1^4 [f(x) - g(x)]\, dx$.

If $f(x) \leq g(x)$ for all x in $[a, b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$

Geometrically, the margin figure illustrates that if f and g are both positive and that $f(x) \leq g(x)$ on the interval $[a, b]$, then the area of region between the graph of f and the x-axis is smaller than the area of region between the graph of g and the x-axis.



Similar sketches for the situations where f or g are sometimes or always negative illustrate that the property holds in other situations as well, but we can avoid all of those different cases using Riemann sums.

If we use the same partition $\mathcal{P}$ and chosen points c_j for Riemann sums for f and g, then $f(c_j) \le g(c_j)$ for each j, so:

$$\sum_{j=1}^{n} f(c_j) \cdot \Delta x_j \le \sum_{j=1}^{n} g(c_j) \cdot \Delta x_j$$

Taking the limit over all such partitions as the mesh of those partitions approaches 0, we get $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

If	$m \le f(x) \le M$ for all x in $[a,b]$
then	$m \cdot (b-a) \le \int_a^b f(x)\,dx \le M \cdot (b-a)$

You may have noticed that we haven't called the justifications of these properties "proofs," in part because we haven't precisely defined what $\lim_{\|\mathcal{P}\| \to 0}$ means, but also because of some other technical details left to more advanced textbooks.

This property follows easily from the previous one. First let $g(x) = M$ so that $f(x) \le M = g(x)$ for all x in $[a,b]$, hence

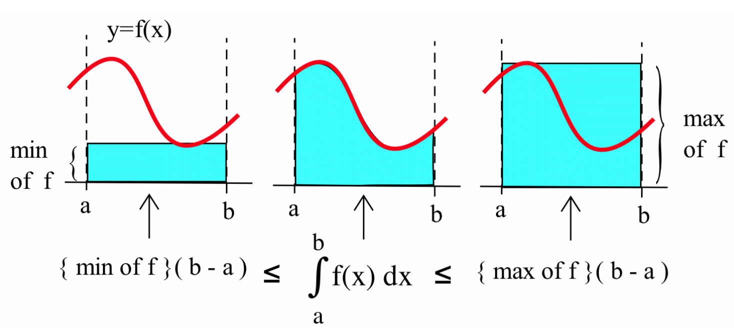
$$\int_a^b f(x)\,dx \le \int_a^b M\,dx = M \cdot (b-a)$$

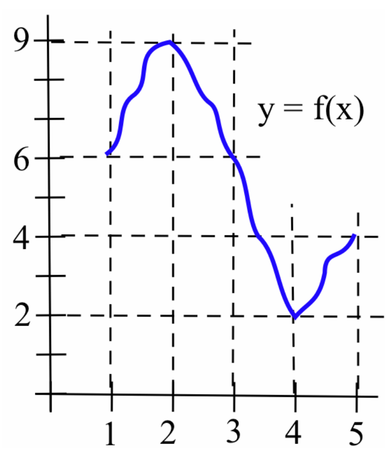
(using one of our previous properties). Likewise, taking $g(x) = m$ so that $f(x) \ge m = g(x)$ for all x in $[a,b]$:

$$\int_a^b f(x)\,dx \ge \int_a^b m\,dx = m \cdot (b-a)$$

Geometrically, this says that if we can "trap" the output values of a function on the interval $[a,b]$ between two upper and lower bounds, m and M, then the value of the definite integral must lie between the areas of the rectangles with heights m and M.

If f is continuous on the closed interval $[a,b]$, then we know that f takes on a minimum value on that interval (call it m) and a maximum value (call it M), in which case this property just uses the lower and upper Riemann sums for the simplest possible partition of $[a,b]$:





Example 1. Determine lower and upper bounds for the value of $\int_1^5 f(x)\,dx$ with $f(x)$ given graphically in the margin.

Solution. If $1 \le x \le 5$, then we can estimate (from the graph) that $2 \le f(x) \le 9$ so a lower bound for $\int_1^5 f(x)\,dx$ is

$$(b-a)\cdot(\text{minimum of } f \text{ on } [a,b]) = (4)(2) = 8$$

and an upper bound is:

$$(b-a)\cdot(\text{maximum of } f \text{ on } [a,b]) = (4)(9) = 36$$

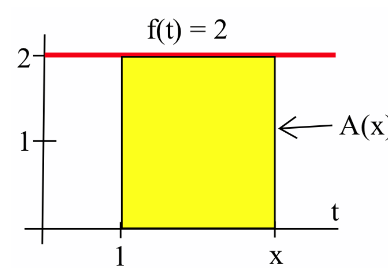
We can conclude that $8 \le \int_1^5 f(x)\,dx \le 36$. ◀

Knowing that the value of a definite integral is somewhere between 8 and 36 is not useful for finding its exact value, but the preceding estimation property is very easy to use and provides a "ballpark estimate" that will help you avoid reporting an unreasonable value.

Practice 2. Determine a lower bound and an upper bound for the value of $\int_3^5 f(x)\,dx$ with f as in the previous Example.

Functions Defined by Integrals

If one of the endpoints a or b of the interval $[a,b]$ changes, then the value of the integral $\int_a^b f(t)\,dt$ typically changes. A definite integral of the form $\int_a^x f(t)\,dt$ defines a function of x that possesses interesting and useful properties. The next examples illustrate one such property: the derivative of a function defined by an integral is closely related to the integrand, the function "inside" the integral.



Example 2. For the function $f(t) = 2$, define $A(x)$ to be the area of the region bounded by f, the t-axis, and vertical lines at $t = 1$ and $t = x$.

(a) Evaluate $A(1)$, $A(2)$, $A(3)$ and $A(4)$.

(b) Find an algebraic formula for $A(x)$ valid for $x \ge 1$.

(c) Calculate $A'(x)$.

(d) Express $A(x)$ as a definite integral.

Solution. (a) Referring to the graph in the margin, we can see that $A(1) = 0$, $A(2) = 2$, $A(3) = 4$ and $A(4) = 6$. (b) Using the same area idea to compute a more general area:

$$A(x) = \text{area of a rectangle} = (\text{base})\,(\text{height}) = (x-1)(2) = 2x-2$$

(c) $A'(x) = \frac{d}{dx}(2x-2) = 2$ (d) $A(x) = \int_1^x 2\,dt$ ◀

Practice 3. Answer the questions in the previous Example for $f(x) = 3$.

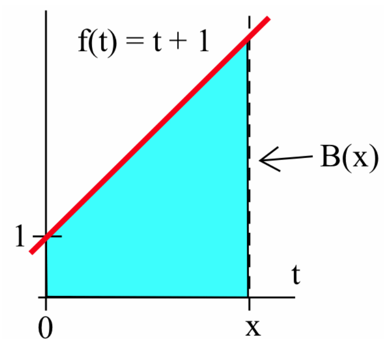
Example 3. For the function $f(t) = 1 + t$, define $B(x)$ to be the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = 0$ and $t = x$ (see margin).

(a) Evaluate $B(0)$, $B(1)$, $B(2)$ and $B(3)$.

(b) Find an algebraic formula for $B(x)$ valid for $x \geq 0$.

(c) Calculate $B'(x)$.

(d) Express $B(x)$ as a definite integral.



Solution. (a) From the graph, $B(0) = 0$, $B(1) = 1.5$, $B(2) = 4$ and $B(3) = 7.5$. (b) Using the same area concept:

$$B(x) = \text{area of trapezoid} = (\text{base}) \cdot (\text{average height})$$
$$= (x) \cdot \left(\frac{1 + (1 + x)}{2} \right) = x + \frac{1}{2}x^2$$

(c) $B'(x) = \frac{d}{dx}\left(x + \frac{1}{2}x^2\right) = 1 + x$ (d) $B(x) = \int_0^x [1 + t]\, dt$ ◀

Practice 4. Answer the questions in the previous Example for $f(t) = 2t$.

A curious "coincidence" appeared in each of these Examples and Practice problems: the derivative of the function defined by the integral was the same as the integrand, the function "inside" the integral. Stated another way, the function defined by the integral was an "antiderivative" of the function "inside" the integral. In Section 4.4 we will see that this "coincidence" is actually a property shared by all functions defined by an integral in this way. And it is such an important property that it is part of a result called the Fundamental Theorem of Calculus. Before we study the Fundamental Theorem of Calculus, however, we need to consider an "existence" question: Which functions can be integrated?

Which Functions Are Integrable?

This important question was finally answered in the 1850s by Bernhard Riemann, a name that should be familiar to you by now. Riemann proved that a function must be *badly* discontinuous in order to not be integrable.

Theorem: Every continuous function is integrable.

Due to our inexact definition of the limit involved in the definition of the definite integral, we defer a proof of this theorem to more advanced textbooks.

This result says that if f is continuous on the interval $[a, b]$, then $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ approaches the same finite number, $\int_a^b f(x)\, dx$, as $\|\mathcal{P}\| \to 0$, no matter how we choose the partitions $\mathcal{P}$.

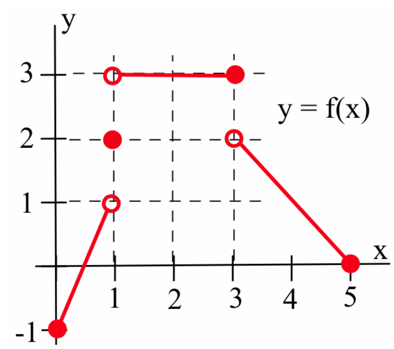
In fact, we can generalize this result to functions that have a finite number of breaks or jumps, as long as the function is bounded:

Theorem:

If f is defined on an interval $[a,b]$ and bounded
($|f(x)| \leq M$ for some number M for all x in $[a,b]$)
and continuous except at a finite number of points in $[a,b]$

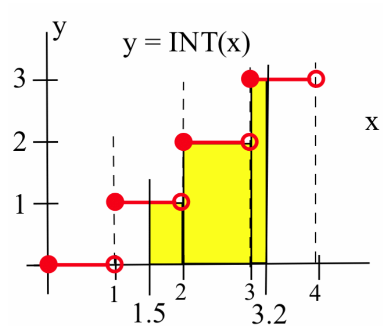
then f is integrable on $[a,b]$.

The function f graphed in the margin is always between -3 and 3 (in fact, always between -1 and 3), so it is bounded, and it is continuous except at $x=1$ and $x=3$. As long as the values of $f(1)$ and $f(3)$ are finite numbers, their actual values will not affect the value of the definite integral, and we can compute the value of the integral by computing the areas of the (triangular and rectangular) regions between the graph of f and the x-axis:

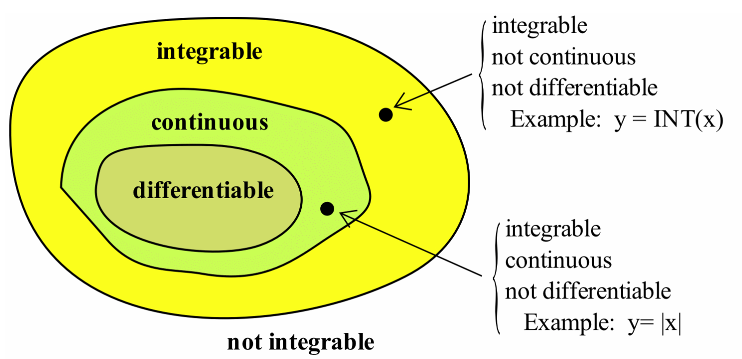


$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = 0+6+2=8$$

Practice 5. Evaluate $\int_{1.5}^{3.2} \lfloor x \rfloor \,dx$ (see margin).



The figure below depicts graphically the relationships between differentiable, continuous and integrable functions:



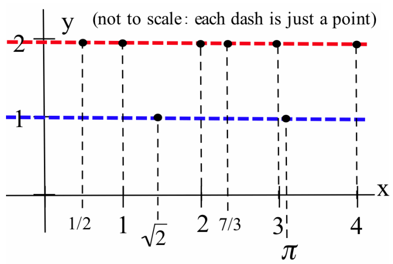
This says:

- Every differentiable function is continuous, but there are continuous functions that are not differentiable: a simple example of the latter is $f(x) = |x|$, which is continuous but not differentiable at $x=0$.
- Every continuous function is integrable, but there are integrable functions that are not continuous: a simple example of the latter situation is the function $f(x)$ graphed in the margin, which is integrable on $[0,5]$ but discontinuous at $x=2$ and $x=3$.
- Finally, as demonstrated by the next example, there are functions that are not integrable.

A Non-integrable Function

If f is continuous or piecewise continuous on $[a, b]$, then f is integrable on $[a, b]$. Fortunately, nearly all of the functions we will use throughout the rest of this book are integrable, as are the functions you are likely to need for common applications.

There are functions, however, for which the limit of the Riemann sums does not exist and hence, by definition, are not integrable. Recall the "holey" function from Section 0.4:



> The function
>
> $$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$
>
> is **not** integrable on $[0, 3]$.

Proof. For any partition $\mathcal{P}$ of $[0, 3]$, suppose that you, a very rational person, always choose values of c_k that are rational numbers. (Any open interval on the real-number line contains rational numbers and irrational numbers, so for each subinterval of the partition $\mathcal{P}$ you can always choose c_k to be a rational number.)

Then $h(c_k) = 2$, so for your Riemann sum:

$$\text{YS}_{\mathcal{P}} = \sum_{k=1}^{n} h(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} 2 \cdot \Delta x_k = 2 \cdot \sum_{k=1}^{n} \Delta x_k = 2 \cdot (3 - 0) = 6$$

Suppose your friend, however, always selects values of c_k that are irrational numbers. Then $h(c_k) = 1$ for each c_k, so for your friend's Riemann sum:

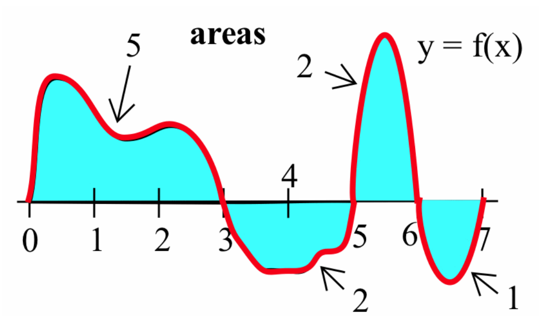
$$\text{FS}_{\mathcal{P}} = \sum_{k=1}^{n} h(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} 1 \cdot \Delta x_k = 1 \cdot \sum_{k=1}^{n} \Delta x_k = 1 \cdot (3 - 0) = 3$$

So the limit of your Riemann sums, as the mesh of $\mathcal{P}$ approaches 0, will be 6, while the limit of your friend's sums will be 3. This means that $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} h(c_k) \cdot \Delta x_k \right)$ does not exist (because there is no single limiting value of the Riemann sums as $\|\mathcal{P}\| \to 0$) so $h(x)$ is not integrable on $[0, 3]$. □

A similar argument shows that $h(x)$ is not integrable on *any* interval of the form $[a, b]$ (where $a < b$).

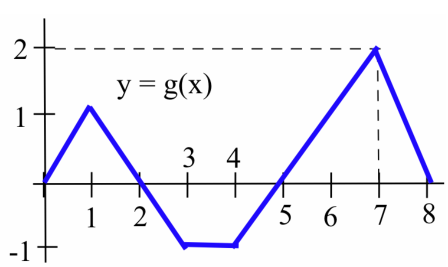
4.3 Problems

In Problems 1–20, refer to the graph of f given below to determine the value of each definite integral.



1. $\int_0^3 f(x)\,dx$
2. $\int_3^5 f(x)\,dx$
3. $\int_2^2 f(x)\,dx$
4. $\int_6^7 f(w)\,dw$
5. $\int_0^5 f(x)\,dx$
6. $\int_0^7 f(x)\,dx$
7. $\int_3^6 f(t)\,dt$
8. $\int_5^7 f(x)\,dx$
9. $\int_3^0 f(x)\,dx$
10. $\int_5^3 f(x)\,dx$
11. $\int_6^0 f(x)\,dx$
12. $\int_0^3 2\cdot f(x)\,dx$
13. $\int_4^4 f^2(s)\,ds$
14. $\int_0^3 [1+f(x)]\,dx$
15. $\int_0^3 [x+f(x)]\,dx$
16. $\int_3^5 [3+f(x)]\,dx$
17. $\int_0^5 [2+f(x)]\,dx$
18. $\int_3^5 |f(x)|\,dx$
19. $\int_0^5 |f(x)|\,dx$
20. $\int_7^3 [1+|f(x)|]\,dx$

Problems 21–30 refer to the graph of g given below. Use the graph to evaluate each integral.



21. $\int_0^2 g(x)\,dx$
22. $\int_1^3 g(t)\,dt$
23. $\int_0^5 g(x)\,dx$
24. $\int_4^2 g(x)\,dx$
25. $\int_0^8 g(s)\,ds$
26. $\int_1^4 |g(x)|\,dx$
27. $\int_0^3 2\cdot g(t)\,dt$
28. $\int_5^8 [1+g(x)]\,dx$
29. $\int_6^3 g(u)\,du$
30. $\int_0^8 [t+g(t)]\,dt$

For 31–34 , use the constant functions $f(x)=4$ and $g(x)=3$ on the interval $[0,2]$. Calculate the value of each integral and verify that the value obtained in part (a) is **not** equal to the value in part (b).

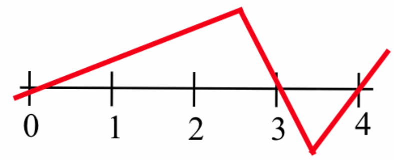
31. (a) $\int_0^2 f(x)\,dx\cdot\int_0^2 g(x)\,dx$ (b) $\int_0^2 f(x)\cdot g(x)\,dx$
32. (a) $\dfrac{\int_0^2 f(x)\,dx}{\int_0^2 g(x)\,dx}$ (b) $\int_0^2 \dfrac{f(x)}{g(x)}\,dx$
33. (a) $\int_0^2 [f(x)]^2\,dx$ (b) $\left(\int_0^2 f(x)\,dx\right)^2$
34. (a) $\int_0^2 \sqrt{f(x)}\,dx$ (b) $\sqrt{\int_0^2 f(x)\,dx}$

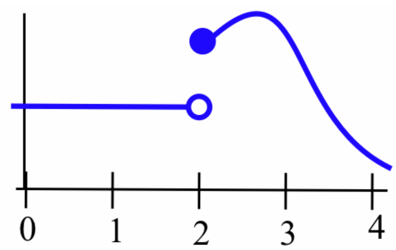
For 35–42, sketch a graph of the integrand function and use it to help evaluate the integral.

35. $\int_0^4 |x|\,dx$
36. $\int_0^4 [1+|t|]\,dt$
37. $\int_{-1}^2 |x|\,dx$
38. $\int_0^2 [|x|-1]\,dx$
39. $\int_1^3 \lfloor u\rfloor\,du$
40. $\int_1^{3.5} \lfloor x\rfloor\,dx$
41. $\int_1^3 [2+\lfloor t\rfloor]\,dt$
42. $\int_3^1 \lfloor x\rfloor\,dx$

For Problems 43–46, sketch (a) a graph of $y = A(x) = \int_0^x f(t)\,dt$ and (b) a graph of $y = A'(x)$.

43. $f(x)=x$
44. $f(x)=x-2$

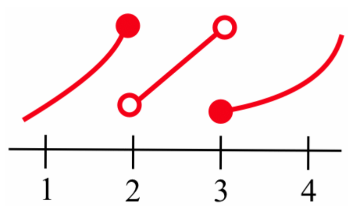
45. 

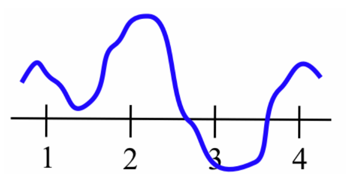
46. 

For 47–50, state whether or not each function is: (a) continuous on $[1,4]$ (b) differentiable on $[1,4]$ (c) integrable on $[1,4]$

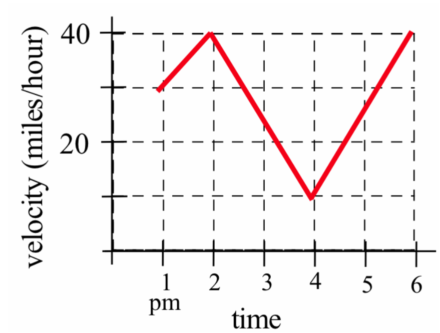
47. $f(x)$ from Problem 45.

48. $f(x)$ from Problem 46.

49. 

50. 

51. The figure below shows the velocity of a car. Write the total distance traveled by the car between 1:00 p.m. and 4:00 p.m. as a definite integral and estimate the value of that integral.



52. Write the total distance traveled by the car in the previous problem between 3:00 p.m. and 6:00 p.m. as a definite integral and estimate the value of that integral.

53. Define $g(x) = 7$ for $x \neq 2$ and $g(2) = 5$.

(a) Show that the Riemann sum for $g(x)$ for any partition $\mathcal{P}$ of the interval $[1,4]$ is equal to $5w + 7(3-w)$, where w is the width of the subinterval that includes $x = 2$.

(b) Compute the limit of these sums, as $\|\mathcal{P}\| \to 0$

(c) Compare the values of $\int_1^4 g(x)\,dx$ and $\int_1^4 7\,dx$.

(d) What can you conclude about how changing the value of an integrable function at a single point affects the value of its definite integral?

4.3 Practice Answers

1. $\displaystyle\int_1^4 [f(x) - g(x)]\,dx = 7 - 3 = 4$

2. $m = 2$ and $M = 6$ so $(2)(5-3) = 4 \leq \displaystyle\int_3^5 f(x)\,dx \leq 12 = (6)(5-3)$

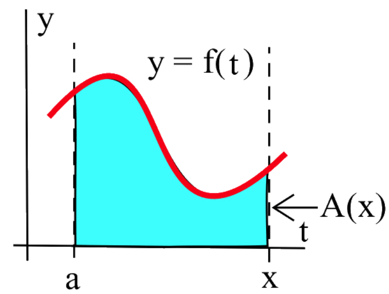
3. (a) $A(1) = 0$, $A(2) = 3$, $A(3) = 6$, $A(4) = 9$
(b) $A(x) = (x-1)(3) = 3x - 3$ (c) $A'(x) = 3$ (d) $A(x) = \displaystyle\int_1^x 3\,dt$

4. (a) $B(0) = 0$, $B(1) = 1$, $B(2) = 4$, $B(3) = 9$
(b) $B(x) = \frac{1}{2}(x)(2x) = x^2$ (c) $B'(x) = 2x$ (d) $A(x) = \displaystyle\int_1^x 2t\,dt$

5. $(0.5)(1) + (1)(2) + (0.2)(3) = 3.1$

4.4 *Areas, Integrals and Antiderivatives*

This section explores properties of functions defined as areas and examines some connections among areas, integrals and antiderivatives. In order to focus on these connections and their geometric meaning, all of the functions in this section are nonnegative, but in the next section we will generalize (and prove) the results for all continuous functions. This section also introduces examples showing how you can use the relationships between areas, integrals and antiderivatives in various applications.



When f is a continuous, nonnegative function, the "area function" $A(x) = \int_a^x f(t)\,dt$ represents the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = a$ and $t = x$ (see margin figure), and the derivative of $A(x)$ represents the rate of change (growth) of $A(x)$ as the vertical line $t = x$ moves rightward. Examples 2 and 3 of Section 4.3 showed that for certain functions f, $A'(x) = f(x)$ so that $A(x)$ was an antiderivative of $f(x)$. The next theorem says the result is true for every continuous, nonnegative function f.

The Area Function Is an Antiderivative

If f is a continuous, nonnegative function

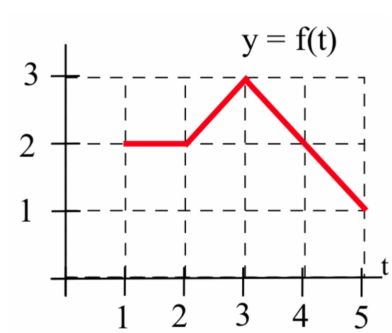
and $A(x) = \int_a^x f(t)\,dt$ for $x \geq a$

then $\dfrac{d}{dx}\left(\int_a^x f(t)\,dt\right) = A'(x) = f(x)$

so $A(x)$ is an antiderivative of $f(x)$.

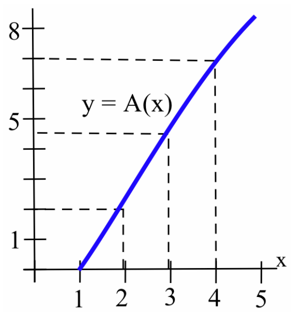
This result relating integrals and antiderivatives is a special case (for nonnegative functions f) of the first part of the Fundamental Theorem of Calculus (FTC[1]), which we will prove in Section 4.5. This result is important for two reasons:

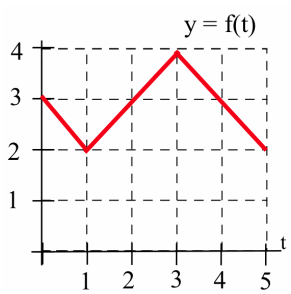
- It says that a large collection of functions have antiderivatives.
- It leads to an **easy** way to **exactly** evaluate definite integrals.



Example 1. Define $A(x) = \int_1^x f(t)\,dt$ for the function $f(t)$ shown in the margin. Estimate the values of $A(x)$ and $A'(x)$ for $x = 2, 3, 4$ and 5 and use these values to sketch a graph of $y = A(x)$.

Solution. Dividing the region into squares and triangles, it is easy to see that $A(2) = 2$, $A(3) = 4.5$, $A(4) = 7$ and $A(5) = 8.5$. Because $A'(x) = f(x)$, we know that $A'(2) = f(2) = 2$, $A'(3) = f(3) = 3$, $A'(4) = f(4) = 2$ and $A'(5) = f(5) = 1$. A graph of $y = A(x)$ appears in the margin at the top of the next page. ◀

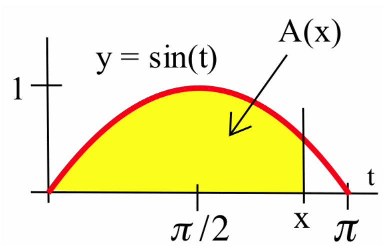


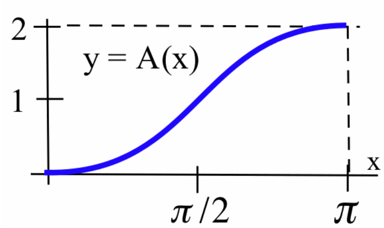


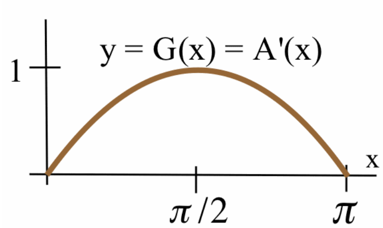
It is important to recognize that f is not differentiable at $x = 2$ or $x = 3$ but that the values of A change smoothly near $x = 2$ and $x = 3$, and the function A *is* differentiable at those points and at every other point between $x = 1$ and $x = 5$. Also note that $f'(4) = -1$ (f is clearly decreasing near $x = 4$) but that $A'(4) = f(4) = 2$ is positive (the area A is growing even though f is getting smaller).

Practice 1. Let $B(x)$ be the area bounded by the horizontal axis, vertical lines at $t = 0$ and $t = x$, and the graph of $f(t)$ shown in the margin. Estimate the values of $B(x)$ and $B'(x)$ for $x = 1, 2, 3, 4$ and 5.

Example 2. Let $G(x) = \frac{d}{dx}\left(\int_0^x \sin(t)\,dt\right)$. Evaluate $G(x)$ for $x = \frac{\pi}{4}$, $\frac{\pi}{2}$ and $\frac{3\pi}{4}$.







Solution. The middle margin figure shows $A(x) = \int_0^x \sin(t)\,dt$ graphically. By the theorem, $A'(x) = \sin(x)$, so:

$$G\left(\frac{\pi}{4}\right) = A'\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.707$$
$$G\left(\frac{\pi}{2}\right) = A'\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$
$$G\left(\frac{3\pi}{4}\right) = A'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.707$$

The penultimate margin figure shows a graph of $y = A(x)$ and the bottom margin figure shows the graph of $y = A'(x) = G(x)$. ◀

Using Antiderivatives to Evaluate $\int_a^b f(x)\,dx$

Now we combine the ideas of areas and antiderivatives to devise a technique for evaluating definite integrals that is exact—and often easy.

If $A(x) = \int_a^x f(t)\,dt$, then we know that $A(a) = \int_a^a f(t)\,dt = 0$, $A(b) = \int_a^b f(t)\,dt$ and that $A(x)$ is an antiderivative of f, so $A'(x) = f(x)$. We also know that if $F(x)$ is **any** antiderivative of f, then $F(x)$ and $A(x)$ have the same derivative so $F(x)$ and $A(x)$ are "parallel" functions and differ by a constant: $F(x) = A(x) + C$ for all x and some constant C. As a consequence:

$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) - A(a)$$
$$= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt$$

This result says that, to evaluate a definite integral $A(b) = \int_a^b f(t)\,dt$, we can find **any** antiderivative F of f and simply evaluate $F(b) - F(a)$.

This result is a special case of the second part of the Fundamental Theorem of Calculus (FTC^2, stated and proved in Section 4.5), which you will use hundreds of times over the next several chapters.

Antiderivatives and Definite Integrals

If f is a continuous, nonnegative function and F is any antiderivative of f (so that $F'(x) = f(x)$) on $[a, b]$

then $\int_a^b f(t)\,dt = F(b) - F(a)$

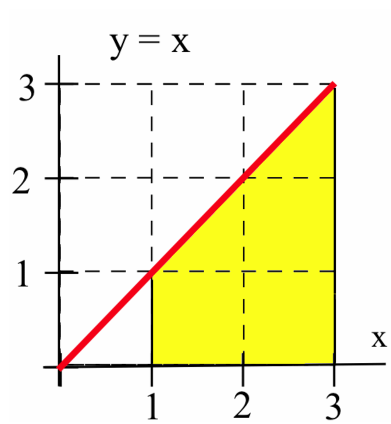
The problem of finding the exact value of a definite integral has been reduced to finding some (any) antiderivative F of the integrand and then evaluating $F(b) - F(a)$. Even finding one antiderivative can be difficult, so for now we will restrict our attention to functions that have "easy" antiderivatives. Later we will explore some methods for finding antiderivatives of more "difficult" functions.

Because an evaluation of the form $F(b) - F(a)$ will occur quite often, we represent it symbolically as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

Example 3. Evaluate $\int_1^3 x\,dx$ in two ways:

(a) by sketching a graph of $y = x$ and finding the area represented by the definite integral.

(b) by finding an antiderivative $F(x)$ of $f(x) = x$ and evaluating $F(3) - F(1)$.



Solution. (a) A graph of $y = x$ appears in the margin; the area of the trapezoidal region in question has area 4. (b) One antiderivative of x is $F(x) = \frac{1}{2}x^2$ (you should check for yourself that $\mathbf{D}\left(\frac{x^2}{2}\right) = x$), so:

$$F(x)\Big|_1^3 = F(3) - F(1) = \frac{1}{2}(3)^2 - \frac{1}{2}(1)^2 = \frac{9}{2} - \frac{1}{2} = 4$$

which agrees with the area from part (a).

If someone chose another antiderivative of x, say $F(x) = \frac{1}{2}x^2 + 7$ (you should check for yourself that $\mathbf{D}\left(\frac{x^2}{2} + 7\right) = x$), then:

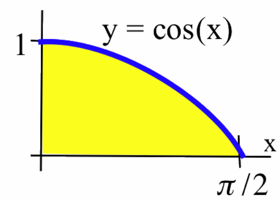
$$F(x)\Big|_1^3 = F(3) - F(1) = \left[\frac{1}{2}(3)^2 + 7\right] - \left[\frac{1}{2}(1)^2 + 7\right] = \frac{23}{2} - \frac{15}{2} = 4$$

No matter which antiderivative F we choose, $F(3) - F(1) = 4$. ◀

Practice 2. Evaluate $\int_1^3 (x-1)\,dx$ in the two ways specified in the previous Example.

This antiderivative method provides an extremely powerful way to evaluate some definite integrals, and we will use it often.

Example 4. Find the area of the region in the first quadrant bounded by the graph of $y = \cos(x)$, the horizontal axis, and the line $x = 0$.



Solution. The area we want (see margin) is $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$ so we need an antiderivative of $f(x) = \cos(x)$. $F(x) = \sin(x)$ is one such antiderivative (you should check that $\mathbf{D}(\sin(x)) = \cos(x)$), so

$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \sin(x)\Big|_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$$

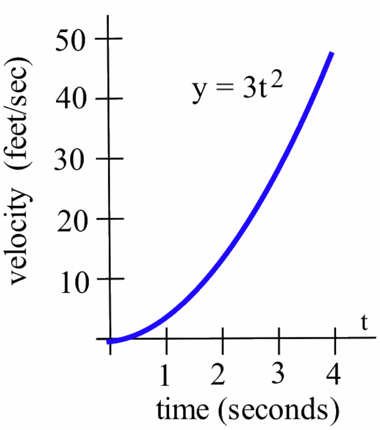
is the area of the region in question. ◀

Practice 3. Find the area of the region bounded by the graph of $y = 3x^2$, the horizontal axis and the vertical lines $x = 1$ and $x = 2$.

Integrals, Antiderivatives and Applications

The antiderivative method for evaluating definite integrals can also be used when we need to find a more general "area," so it is often useful for solving applied problems.

Example 5. A robot has been programmed so that when it starts to move, its velocity after t seconds will be $3t^2$ feet per second.



(a) How far will the robot travel during its first four seconds of movement?

(b) How far will the robot travel during its next four seconds of movement?

(c) How long will it take for the robot to move 729 feet from its starting place?

Solution. (a) The distance during the first four seconds will be the area under the graph of the velocity function (see margin figure) from $t = 0$ to $t = 4$, an area we can compute with the definite integral $\int_0^4 3t^2\,dt$. One antiderivative of $3t^2$ is t^3 so:

$$\int_0^4 3t^2\,dt = \left[t^3\right]_0^4 = 4^3 - 0^3 = 64$$

and we can conclude that the robot will be 64 feet away from its starting position after four seconds.

(b) Proceeding similarly:

$$\int_4^8 3t^2\,dt = \left[t^3\right]_4^8 = 8^3 - 4^3 = 512 - 64 = 448 \text{ feet}$$

(c) This question is different from the first two. Here we know the lower integration endpoint, $t = 0$, and the total distance, 729 feet, and need to find the upper integration endpoint (the time when the robot is 729 feet away from its starting position). Calling this upper endpoint T, we know that:

$$729 = \int_0^T 3t^2\,dt = \left[t^3\right]_0^T = T^3 - 0^3 = T^3$$

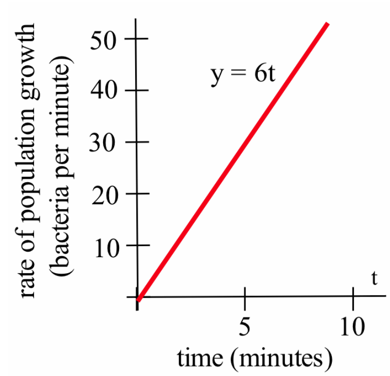
so $T = \sqrt[3]{729} = 9$. The robot is 729 feet away after 9 seconds. ◀

Practice 4. Refer to the robot from the previous Example.

(a) How far will the robot travel between $t = 1$ and $t = 5$ seconds?

(b) How long will it take for the robot to move 343 feet from its starting place?

Example 6. Suppose that t minutes after placing 1,000 bacteria on a Petri plate the rate of growth of the bacteria population is $6t$ bacteria per minute.



(a) How many new bacteria are added to the population during the first seven minutes?

(b) What is the total population after seven minutes?

(c) When will the total population reach 2,200 bacteria?

Solution. (a) The number of new bacteria is represented by the area under the rate-of-growth graph (see margin) and one antiderivative of $6t$ is $3t^2$ (check that $\mathbf{D}\left(3t^2\right) = 6t$) so:

$$\text{new bacteria} = \int_0^7 6t\,dt = \left[3t^2\right]_0^7 = 3(7)^2 - 3(0)^2 = 147$$

(b) [old population] + [new bacteria] $= 1000 + 147 = 1147$ bacteria.

(c) When the total population reaches 2,200 bacteria, then there are $2200 - 1000 = 1200$ new bacteria, hence we need to find the time T required for that many new bacteria to grow:

$$1200 = \int_0^T 6t\,dt = \left[3t^2\right]_0^T = 3(T)^2 - 3(0)^2 = 3T^2$$

so $T^2 = 400 \Rightarrow T = 20$. After 20 minutes, the total bacteria population will be $1000 + 1200 = 2200$. ◀

Practice 5. Refer to the bacteria population from the previous Example.

(a) How many new bacteria will be added to the population between $t = 4$ and $t = 8$ minutes?

(b) When will the total population reach 2,875 bacteria?

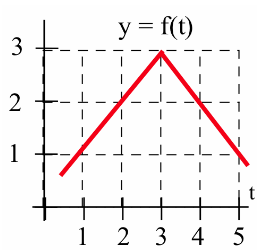
4.4 Problems

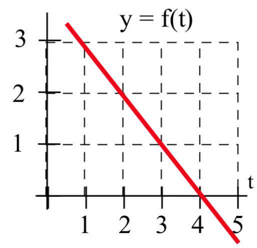
In Problems 1–8, $A(x) = \int_1^x f(t)\,dt$ with $f(t)$ given.

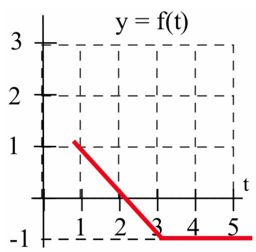
(a) Graph $y = A(x)$ for $1 \le x \le 5$.

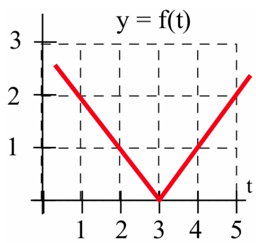
(b) Estimate the values of $A(1)$, $A(2)$, $A(3)$ and $A(4)$.

(c) Estimate $A'(1)$, $A'(2)$, $A'(3)$ and $A'(4)$.

1. 

2. 

3. 

4. 

5. $f(t) = 2$

6. $f(t) = 1 + t$

7. $f(t) = 6 - t$

8. $f(t) = 1 + 2t$

In Problems 9–18, use the **Antiderivatives and Definite Integrals** Theorem to evaluate each integral.

9. (a) $\int_0^3 2x\,dx$ (b) $\int_1^3 2x\,dx$ (c) $\int_0^1 2x\,dx$

10. (a) $\int_0^2 4x^3\,dx$ (b) $\int_0^1 4x^3\,dx$ (c) $\int_1^2 4x^3\,dx$

11. (a) $\int_1^3 6x^2\,dx$ (b) $\int_1^2 6x^2\,dx$ (c) $\int_0^3 6x^2\,dx$

12. (a) $\int_{-2}^2 2x\,dx$ (b) $\int_{-2}^{-1} 2x\,dx$ (c) $\int_{-2}^0 2x\,dx$

13. (a) $\int_0^3 4x^3\,dx$ (b) $\int_1^3 4x^3\,dx$ (c) $\int_0^1 4x^3\,dx$

14. (a) $\int_0^5 4x^3\,dx$ (b) $\int_0^2 4x^3\,dx$ (c) $\int_2^5 4x^3\,dx$

15. (a) $\int_{-3}^3 3x^2\,dx$ (b) $\int_{-3}^0 3x^2\,dx$ (c) $\int_0^3 3x^2\,dx$

16. (a) $\int_0^3 5\,dx$ (b) $\int_0^2 5\,dx$ (c) $\int_2^3 5\,dx$

17. (a) $\int_0^2 3x^2\,dx$ (b) $\int_1^3 3x^2\,dx$ (c) $\int_3^1 3x^2\,dx$

18. (a) $\int_{-2}^2 \left[12 - 3x^2\right]\,dx$ (b) $\int_1^2 \left[12 - 3x^2\right]\,dx$

In 19–21, use the given velocity of a car (in feet per second) after t seconds to find:

(a) how far the car travels during the first 10 seconds.

(b) how many seconds it takes the car to travel half the distance in part (a).

19. $v(t) = 2t$ 20. $v(t) = 3t^2$ 21. $v(t) = 4t^3$

Problems 22–23 give the velocity of a car (in feet per second) after t seconds.

(a) How many seconds does it take for the car to come to a stop (velocity $= 0$)?

(b) How far does the car travel before coming to a stop?

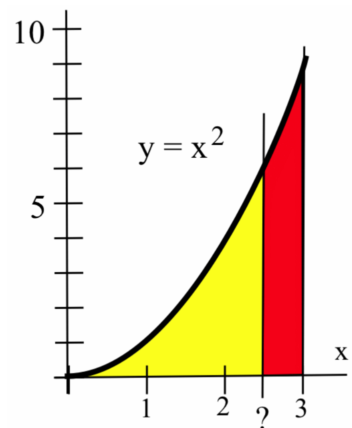
(c) How many seconds does it take the car to travel half the distance in part (b)?

22. $v(t) = 20 - 2t$ 23. $v(t) = 75 - 3t^2$

24. Find the exact area under half of one arch of the sine curve: $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$.

25. An artist you know wants to make a figure consisting of the region between the curve $y = x^2$ and the x-axis for $0 \le x \le 3$.

    (a) Where should the artist divide the region with a vertical line (see figure below) so that each piece has the same area?



    (b) Where should she divide the region with vertical lines to get three pieces with equal areas?

4.4 Practice Answers

1. $B(1) = 2.5$, $B(2) = 5$, $B(3) = 8.5$, $B(4) = 12$, $B(5) = 14.5$
$B(x) = \int_0^x f(t)\,dt \Rightarrow B'(x) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$
(by the **Area Function Is an Antiderivative** Theorem), hence:
$B'(1) = f(1) = 2$, $B'(2) = f(2) = 3$, $B'(3) = 4$, $B'(4) = 3$ and $B'(5) = 2$.

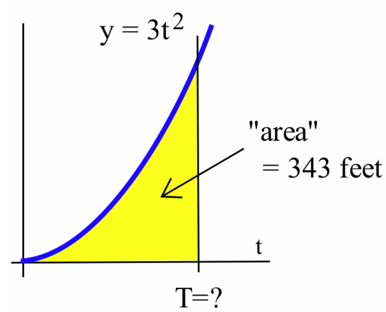
2. (a) $\int_1^3 (x-1)\,dx$ gives the area of the triangular region between the graph of $y = x - 1$ and the x-axis for $1 \le x \le 3$:

$$\text{area} = \frac{1}{2}\,(\text{base})\,(\text{height}) = \frac{1}{2}(2)(2) = 2$$

(b) $F(x) = \frac{1}{2}x^2 - x$ is an antiderivative of $f(x) = x - 1$ so:

$$\int_1^3 (x-1)\,dx = F(3) - F(1) = \left[\frac{1}{2}\cdot 3^3 - 3\right] - \left[\frac{1}{2}\cdot 1^3 - 1\right] = 2$$

3. Area $= \int_1^2 3x^2\,dx = x^3\Big|_1^2 = 2^3 - 1^3 = 8 - 1 = 7$



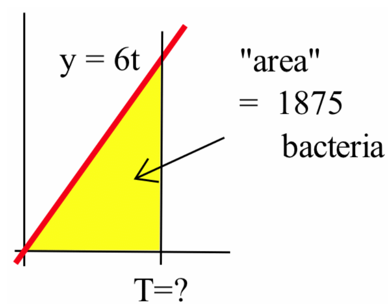
4. (a) distance $= \int_1^5 3t^2\,dt = t^3\Big|_1^5 = 125 - 1 = 124$ feet.

(b) We know the starting point is $x = 0$ and the total distance ("area" under the velocity curve) is 343 feet. We need to find the time T (see margin figure) so that 343 feet $= \int_0^T 3t^2\,dt$:

$$343 = \int_0^T 3t^2\,dt = t^3\Big|_0^T = T^3 - 0 = T^3$$

hence $T = \sqrt[3]{343} = 7$ seconds.

5. (a) new bacteria $= \int_4^8 6t\,dt = 3t^2\Big|_4^8 = 3\cdot 64 - 3\cdot 16 = 144$ bacteria.



(b) We know the total new population ("area" under the rate-of-change graph) is $2875 - 1000 = 1875$ so:

$$1875 = \int_0^T 6t\,dt = 3t^2\Big|_0^T = 3T^2 - 0 = 3T^2 \Rightarrow T^2 = 625$$

hence $T = \sqrt{625} = 25$ minutes.

4.5 The Fundamental Theorem of Calculus

This section contains the most important and most frequently used theorem of calculus, **THE** Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz during the late 1600s, it establishes a connection between derivatives and integrals, provides a way to easily calculate many definite integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is the most important brick in that beautiful structure.

Prior sections have emphasized the meaning of the definite integral, defined it, and began to explore some of its applications and properties. In this section, the emphasis shifts to the Fundamental Theorem of Calculus. You will use this theorem often in later sections.

The Fundamental Theorem has two parts. They resemble results in the previous section but apply to more general situations. The first part (FTC^1) says that every continuous function has an antiderivative and shows how to differentiate a function defined as an integral. The second part (FTC^2) shows how to evaluate the definite integral of any function if we know a formula for an antiderivative of that function.

Part 1: Antiderivatives

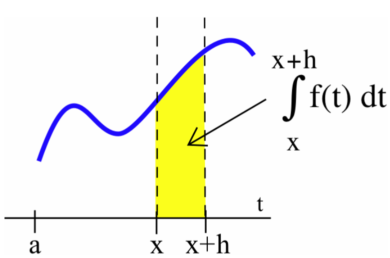
Every continuous function has an antiderivative, even functions with "corners," such as the absolute value function $f(x) = |x|$, that fail to be differentiable at one or more points.

The Fundamental Theorem of Calculus Part 1 (FTC^1)

If f is continuous and $A(x) = \int_a^x f(t)\,dt$

then $A'(x) = \dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

so $A(x)$ is an antiderivative of $f(x)$.



Proof. For a continuous function f, let $A(x) = \int_a^x f(t)\,dt$. By the definition of derivative,

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

Using one of the integral properties from Section 4.3, we know that:

$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$$

$$\Rightarrow \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

Assume for the moment that $h > 0$. Because f is continuous on $[x, x+h]$ we know that f attains a maximum and minimum on that interval, so there are values m_h and M_h with $x < m_h < x+h$ and $x < M_h < x+h$ so that $f(m_h) \le f(t) \le f(M_h)$ when $x \le t \le x+h$. Hence:

$$\int_x^{x+h} f(m_h)\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} f(M_h)\,dt$$

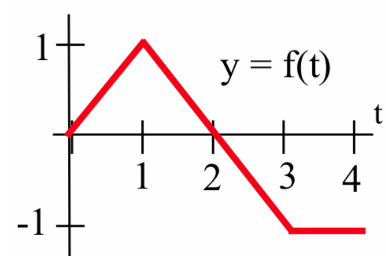
$$\Rightarrow \quad f(m_h)\cdot h \le \int_x^{x+h} f(t)\,dt \le f(M_h)\cdot h$$

$$\Rightarrow \quad f(m_h) \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le f(M_h)$$

Because $x < m_h < x+h$, we know $\lim_{h\to 0^+} m_h = x$; consequently—because $f(t)$ is continuous—we also know that $\lim_{h\to 0^+} f(m_h) = f(x)$. Likewise, $\lim_{h\to 0^+} f(M_h) = f(x)$, so the Squeezing Theorem tells us that:

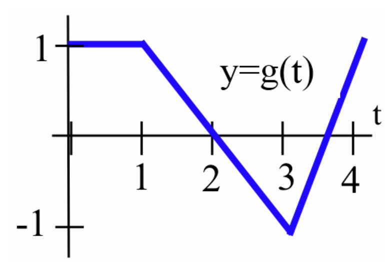
$$\lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$$

Repeating this argument for $h < 0$ is relatively straightforward. □

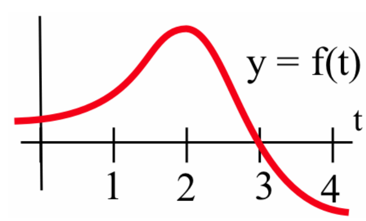


Example 1. Define $A(x) = \int_0^x f(t)\,dt$ for f in the margin figure. Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.

Solution. $A(1) = \int_0^1 f(t)\,dt = \frac{1}{2}$, $A(2) = \int_0^2 f(t)\,dt = 1$, $A(3) = \int_0^3 f(t)\,dt = \frac{1}{2}$ and $A(4) = \int_0^4 f(t)\,dt = -\frac{1}{2}$. Because f is continuous, FTC[1] tells us that $A'(x) = f(x)$, so $A'(1) = f(1) = 1$, $A'(2) = f(2) = 0$, $A'(3) = f(3) = -1$ and $A'(4) = f(4) = -1$. ◀



Practice 1. Define $A(x) = \int_0^x g(t)\,dt$ for g in the margin figure. Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.



Example 2. Define $A(x) = \int_0^x f(t)\,dt$ for f in the margin figure. For which value of x is $A(x)$ maximum? For which x is the rate of change of A maximum?

Solution. Because A is differentiable, the only critical points are where $A'(x) = 0$ or at endpoints. $A'(x) = f(x) = 0$ at $x = 3$, and A has a maximum at $x = 3$. Notice that the values of $A(x)$ increase as x goes from 0 to 3 and then the values of A decrease. The rate of change of $A(x)$ is $A'(x) = f(x)$, and $f(x)$ appears to have a maximum at $x = 2$, so the rate of change of $A(x)$ is maximum when $x = 2$. Near $x = 2$, a slight increase in the value of x yields the maximum increase in the value of $A(x)$. ◀

Part 2: Evaluating Definite Integrals

If we know a formula for an antiderivative of a function, then we can compute any definite integral of that function.

The Fundamental Theorem of Calculus Part 2 (FTC^2)

If $f(x)$ is continuous
and $F(x)$ is any antiderivative of f (so that $F'(x) = f(x)$)

then $\displaystyle\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$.

Proof. Define $A(x) = \int_a^x f(t)\,dt$. If F is an antiderivative of f, then $F'(x) = f(x)$ and by FTC^1 we know that $A'(x) = f(x)$ so $F'(x) = A'(x)$, hence $F(x)$ and $A(x)$ differ by a constant: $A(x) - F(x) = C$ for all x and some constant C. At $x = a$, we have $C = A(a) - F(a) = 0 - F(a) = -F(a)$ so $C = -F(a)$ and the equation $A(x) - F(x) = C$ becomes $A(x) - F(x) = -F(a)$. Then $A(x) = F(x) - F(a)$ for all x, so setting $x = b$ yields $A(b) = F(b) - F(a)$, hence $\int_a^b f(x)\,dx = F(b) - F(a)$, the formula we wanted. □

We can evaluate the definite integral of a continuous function f by finding an antiderivative of f (*any* antiderivative of f will work) and then doing some arithmetic with this antiderivative. FTC^2 does not tell us *how* to find an antiderivative of f, and it does not tell us how to find the definite integral of a discontinuous function. It is possible to evaluate definite integrals of some discontinuous functions (as we saw in Section 4.3) but not by using FTC^2 directly.

Example 3. Evaluate $\int_0^2 \left(x^2 - 1\right)\,dx$.

Solution. $F(x) = \frac{1}{3}x^3 - x$ is an antiderivative of $f(x) = x^2 - 1$ (you should check that $\mathbf{D}\left(\frac{1}{3}x^3 - x\right) = x^2 - 1$), so:

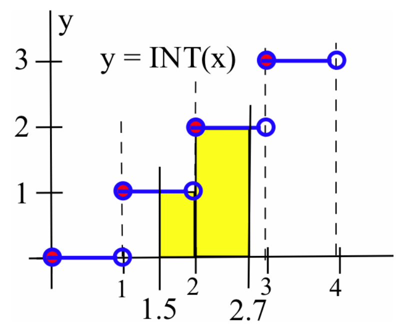
$$\int_0^2 \left(x^2 - 1\right)\,dx = \left[\frac{1}{3}x^3 - x\right]_0^2 = \left[\frac{1}{3}\cdot 2^3 - 2\right] - \left[\frac{1}{3}\cdot 0^3 - 0\right] = \frac{2}{3}$$

If your friend had picked a different antiderivative of $x^2 - 1$, say $G(x) = \frac{1}{3}x^3 - x + 4$, then her calculations would be slightly different :

$$\begin{aligned}\int_0^2 \left(x^2 - 1\right)\,dx &= \left[\frac{1}{3}x^3 - x + 4\right]_0^2 \\ &= \left[\frac{1}{3}\cdot 2^3 - 2 + 4\right] - \left[\frac{1}{3}\cdot 0^3 - 0 + 4\right] = \frac{2}{3} + 4 - 4 = \frac{2}{3}\end{aligned}$$

but the result would be the same. ◄

Practice 2. Evaluate $\int_1^3 \left(3x^2 - 1\right)\, dx$.



Example 4. Evaluate $\int_{1.5}^{2.7} \lfloor x \rfloor\, dx$ (where $\lfloor x \rfloor = \text{INT}(x)$ is the largest integer less than or equal to x, as in the margin figure).

Solution. $f(x) = \lfloor x \rfloor$ is not continuous at $x = 2$ in the interval $[1.5, 2.7]$, so we cannot employ the Fundamental Theorem of Calculus directly. We can, however, use our understanding of the geometric meaning of a definite integral to compute:

$$\begin{aligned}\int_{1.5}^{2.7} \lfloor x \rfloor\, dx &= (\text{area below } y = \lfloor x \rfloor \text{ for } 1.5 \le x \le 2) + (\text{area below } y = \lfloor x \rfloor \text{ for } 2 \le x \le 2.7)\\ &= (\text{first base})\,(\text{first height}) + (\text{second base})\,(\text{second height})\\ &= (0.5)(1) + (0.7)(2) = 1.9\end{aligned}$$

We could also split the integral into two pieces:

$$\begin{aligned}\int_{1.5}^{2.7} \lfloor x \rfloor\, dx &= \int_{1.5}^{2.0} \lfloor x \rfloor\, dx + \int_{2.0}^{2.7} \lfloor x \rfloor\, dx\\ &= \int_{1.5}^{2.0} 1\, dx + \int_{2.0}^{2.7} 2\, dx = \Big[x\Big]_{1.5}^{2.0} + \Big[x\Big]_{2.0}^{2.7}\\ &= [2.0 - 1.5] + [2(2.7) - 2(2.0)] = 0.5 + 1.4 = 1.9\end{aligned}$$

using the fact that $\lfloor x \rfloor = 1$ for $1.5 \le x < 2.0$ and the fact that $\lfloor x \rfloor = 2$ for $2.0 \le x \le 2.7$. (We also need to redefine the first integrand to equal 1 at its right endpoint and the second integrand to equal 2 at its right endpoint so that each integrand is continuous on a closed interval). ◀

Problem 53 in Section 4.3 indicates that this redefinition is perfectly legal.

Practice 3. Evaluate $\int_{1.3}^{3.4} \lfloor x \rfloor\, dx$.

Calculus is the study of derivatives and integrals, their meanings and their applications. The Fundamental Theorem of Calculus demonstrates how differentiation and integration are closely related processes: integration is really anti-differentiation, the inverse of differentiation.

Applications: The Future

Calculus is important for many reasons, but students are usually required to study calculus because they will need to *apply* calculus concepts in a variety of fields. Most applied problems in integral calculus require the following steps to get from a real-life problem to a numerical answer:

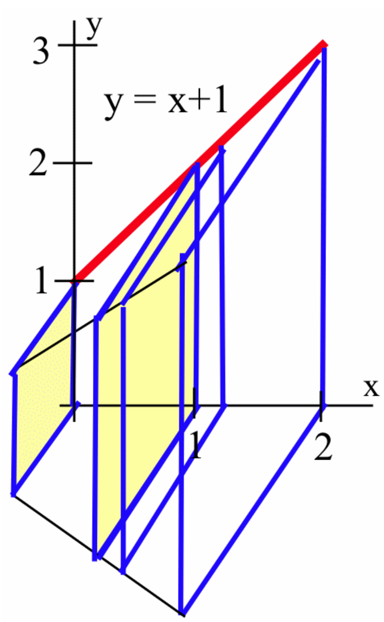
applied problem $\xrightarrow{1}$ Riemann sum $\xrightarrow{2}$ definite integral $\xrightarrow{3}$ number

Step 1 is absolutely vital. If we can not translate the ideas of an applied problem into an area or a Riemann sum or a definite integral, then we can not use integral calculus to solve the problem. For a few special types of applied problems, we will be able to move directly from the problem to an integral, but usually it will be easier to first break the problem into smaller pieces and to build a Riemann sum. Section 4.7 and all of Chapter 5 focus on translating different types of applied problems into Riemann sums and definite integrals. **Computers and calculators are seldom of any help with Step 1.**

Step 2 is usually easy. If we have a Riemann sum $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ on an interval $[a, b]$, then the limit of the sum (as $n \to \infty$) is simply the definite integral $\int_a^b f(x)\,dx$.

Step 3 can be handled in several ways.

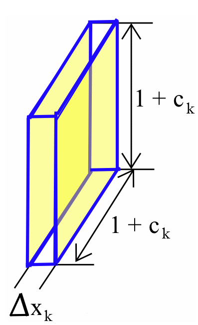
- If the function f is relatively simple, we may be able to find an antiderivative for f (using techniques from Section 4.6 and Chapter 8) and then apply FTC^2 to get a numerical answer.
- If the function f is more complicated, then integral tables or computers (Section 4.8) may help us find an antiderivative for f, in which case we can apply FTC^2 to get a numerical answer.
- If we cannot find an antiderivative for f, we can compute approximate numerical answers for the definite integral using various approximation methods (Sections 4.9 and 8.7); we typically employ computers to carry out the heavy-duty arithmetic.



Usually any difficulties in solving an applied problem arise in the first and third steps. There are techniques and details to master and understand, but it is also important to keep in mind where these techniques and details fit into the bigger picture.

The next Example illustrates these steps for the problem of finding a volume of a solid. We will explore techniques for finding volumes of solids in greater detail in Chapter 5.

Example 5. Find the volume of the solid shown in the margin for $0 \le x \le 2$. (Each "slice" perpendicular to the xy-plane is a square.)



Solution. Step 1: Going from the figure to a Riemann sum.
If we break the solid into n "slices" with cuts perpendicular to the x-axis (and the xy-plane) using a partition $\mathcal{P}$ with cuts at x_1, x_2, x_3,..., x_{n-1} (like slicing a block of cheese or a loaf of bread), then the volume of the original solid is equal to the sum of the volumes of the "slices."

The volume of the k-th slice is *approximately* equal to the volume of a thin, rectangular box:

$$(\text{height}) \cdot (\text{base}) \cdot (\text{thickness}) \approx (c_k + 1)\,(c_k + 1) \cdot \Delta x_k$$

where c_k is any chosen value between x_{k-1} and x_k. Therefore:

$$\text{total volume} = \sum_{k=1}^{n} (\text{volume of the } k\text{-th slice}) = \sum_{k=1}^{n} (c_k+1)^2 \, \Delta x_k$$

which is a Riemann sum.

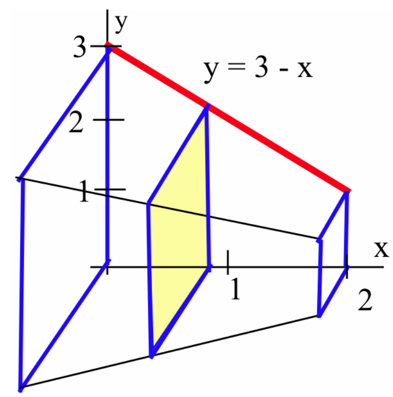
Step 2: Going from the Riemann sum to a definite integral.
We can improve the Riemann sum approximation of the total volume from Step 1 by taking thinner slices (making all of the Δx_k smaller and smaller) so that the mesh of the partition $\mathcal{P}$ approaches 0:

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{k=1}^{n} (c_k+1)^2 \, \Delta x_k = \int_0^2 (x+1)^2 \, dx = \int_0^2 \left[x^2+2x+1\right] \, dx$$

Step 3: Going from the definite integral to a numerical answer.
We can now use FTC^2 to evaluate the integral: $F(x) = \frac{1}{3}x^3 + x^2 + x$ is an antiderivative of x^2+2x+1 (check this by differentiating $F(x)$), so:

$$\int_0^2 \left[x^2+2x+1\right] \, dx = \left[\frac{1}{3}x^3 + x^2 + x\right]_0^2$$
$$= \left[\frac{1}{3}\cdot 2^3 + 2^2 + 2\right] - \left[\frac{1}{3}\cdot 0^3 + 0^2 + 0\right] = \frac{26}{3}$$

The volume of the solid shape is exactly $\frac{26}{3}$ cubic inches. ◀



Practice 4. Find the volume of the solid shape in the margin figure for $0 \le x \le 2$. (Each "slice" perpendicular to the xy-plane is a square.)

Leibniz's Rule For Differentiating Integrals

If the endpoint of an integral is a function of x rather than simply x, then we need to use the Chain Rule together with FTC^1 to calculate the derivative of the integral. For example:

$$A'(x) = f(x) \qquad \Rightarrow \qquad \frac{d}{dx}\left[A\left(x^2\right)\right] = A'(x)\cdot 2x = f\left(x^2\right)\cdot 2x$$

We can generalize this result by applying the Chain Rule to the derivative of the integral:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = \frac{d}{dx}\left[A\left(g(x)\right)\right] = f\left(g(x)\right)\cdot g'(x)$$

and combine this with some integral properties to further extend FTC^1.

Leibniz's Rule

If f is a continuous function, $A(x) = \int_a^x f(t)\,dt$
and $g_1(x)$ and $g_2(x)$ are both differentiable functions

then $\frac{d}{dx}\left[\int_{g_1(x)}^{g_2(x)} f(t)\,dt\right] = f\left(g_2(x)\right)\cdot g_2'(x) - f\left(g_1(x)\right)\cdot g_1'(x)$

Proof. Assume for simplicity that f, g_1 and g_2 are continuous on $(-\infty, \infty)$ and let c be any number. Then:

$$\begin{aligned}\int_{g_1(x)}^{g_2(x)} f(t)\,dt &= \int_c^{g_2(x)} f(t)\,dt + \int_{g_1(x)}^{c} f(t)\,dt \\ &= \int_c^{g_2(x)} f(t)\,dt - \int_c^{g_1(x)} f(t)\,dt\end{aligned}$$

Now apply the preceding result. □

Example 6. If a is any constant, compute the derivatives $\frac{d}{dx}\left[\int_a^{5x} t^2\,dt\right]$, $\frac{d}{dx}\left[\int_a^{x^2} \cos(u)\,du\right]$ and $\frac{d}{dw}\left[\int_{\pi w}^{\sin w} z^3\,dz\right]$.

Solution. Applying Leibniz's Rule:

$$\begin{aligned}\frac{d}{dx}\left[\int_a^{5x} t^2\,dt\right] &= (5x)^2 \cdot 5 = 125x^2 \\ \frac{d}{dx}\left[\int_a^{x^2} \cos(u)\,du\right] &= \cos(x^2) \cdot 2x = 2x\cos(x^2) \\ \frac{d}{dw}\left[\int_{\pi w}^{\sin(w)} z^3\,dz\right] &= (\sin(w))^3 \cdot \cos(w) - (\pi w)^3 \cdot \pi\end{aligned}$$

The last quantity simplifies to $\sin^3(w)\cos(w) - \pi^4 w^3$. ◀

Practice 5. Compute $\frac{d}{dx}\left[\int_0^{x^3} \sin(t)dt\right]$.

4.5 Problems

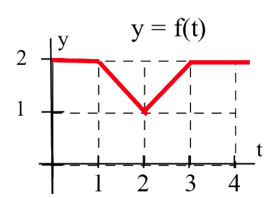
In Problems 1–2, (a) Use FTC[2] to find a formula for $A(x)$, differentiate $A(x)$ to obtain a formula for $A'(x)$, and evaluate $A'(x)$ at $x = 1, 2$ and 3. (b) Use FTC[1] to evaluate $A'(x)$ at $x = 1, 2$ and 3.

1. $A(x) = \int_0^x 3t^2\,dt$
2. $A(x) = \int_1^x (1+2t)\,dt$

In Problems 3–8, compute $A'(1)$, $A'(2)$ and $A'(3)$.

3. $A(x) = \int_0^x 2t\,dt$
4. $A(x) = \int_1^x 2t\,dt$
5. $A(x) = \int_{-3}^x 2t\,dt$
6. $A(x) = \int_0^x \left(3 - t^2\right)\,dt$
7. $A(x) = \int_0^x \sin(t)\,dt$
8. $A(x) = \int_1^x |t-2|\,dt$

In 9–12, $A(x) = \int_0^x f(t)\,dt$, with $f(t)$ given graphically. Evaluate $A'(1)$, $A'(2)$ and $A'(3)$.

9.


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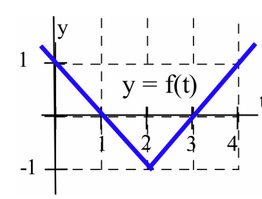

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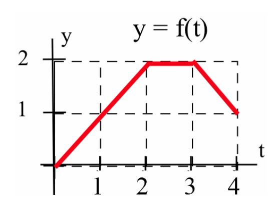

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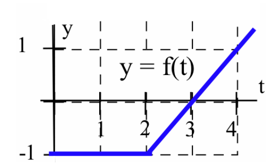


In 13–33, verify that $F(x)$ is an antiderivative of the integrand and use FTC^2 to evaluate the integral.

13. $\int_0^1 2x\,dx, \quad F(x) = x^2 + 5$

14. $\int_1^4 3x^2\,dx, \quad F(x) = x^3 + 2$

15. $\int_1^3 x^2\,dx, \quad F(x) = \frac{1}{3}x^3$

16. $\int_0^3 \left[x^2 + 4x - 3\right]\,dx, F(x) = \frac{1}{3}x^3 + 2x^2 - 3x$

17. $\int_1^5 \frac{1}{x}\,dx, \quad F(x) = \ln(x)$

18. $\int_2^5 \frac{1}{x}\,dx, \quad F(x) = \ln(x) + 4$

19. $\int_{\frac{1}{2}}^3 \frac{1}{x}\,dx, \quad F(x) = \ln(x)$

20. $\int_1^3 \frac{1}{x}\,dx, \quad F(x) = \ln(x) + 2$

21. $\int_0^{\frac{\pi}{2}} \cos(x)\,dx, \quad F(x) = \sin(x)$

22. $\int_0^{\pi} \sin(x)\,dx, \quad F(x) = -\cos(x)$

23. $\int_0^1 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$

24. $\int_1^4 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$

25. $\int_1^7 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$

26. $\int_1^4 \frac{1}{2\sqrt{x}}\,dx, \quad F(x) = \sqrt{x}$

27. $\int_1^9 \frac{1}{2\sqrt{x}}\,dx, \quad F(x) = \sqrt{x}$

28. $\int_2^5 \frac{1}{x^2}\,dx, \quad F(x) = -\frac{1}{x}$

29. $\int_{-2}^3 e^x\,dx, \quad F(x) = e^x$

30. $\int_0^3 \frac{2x}{1+x^2}\,dx, \quad F(x) = \ln(1+x^2)$

31. $\int_0^{\frac{\pi}{4}} \sec^2(x)\,dx, \quad F(x) = \tan(x)$

32. $\int_1^e \ln(x)\,dx, \quad F(x) = x \cdot \ln(x) - x$

33. $\int_0^3 2x\sqrt{1+x^2}\,dx, \quad F(x) = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}$

For 34–48, find an antiderivative of the integrand and use FTC^2 to evaluate the definite integral.

34. $\int_2^5 3x^2\,dx$

35. $\int_{-1}^2 x^2\,dx$

36. $\int_1^3 \left[x^2 + 4x - 3\right]\,dx$

37. $\int_1^e \frac{1}{x}\,dx$

38. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)\,dx$

39. $\int_{25}^{100} \sqrt{x}\,dx$

40. $\int_3^5 \sqrt{x}\,dx$

41. $\int_1^{10} \frac{1}{x^2}\,dx$

42. $\int_1^{1000} \frac{1}{x^2}\,dx$

43. $\int_0^1 e^x\,dx$

44. $\int_{-2}^2 \frac{2x}{1+x^2}\,dx$

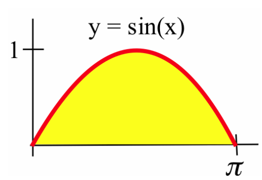
45. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2(x)\,dx$

46. $\int_0^1 e^{2x}\,dx$

47. $\int_3^3 \sin(x) \cdot \ln(x)\,dx$

48. $\int_2^4 (x-2)^3\,dx$

In 49–54, find the area of the shaded region.

49.


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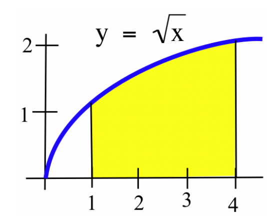

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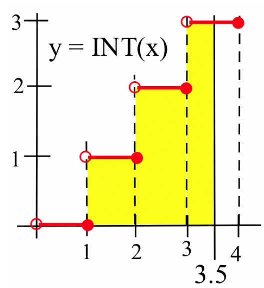

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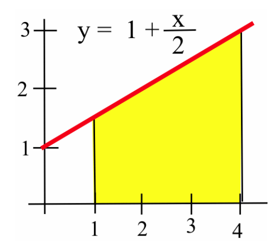

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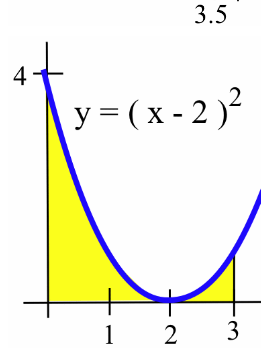

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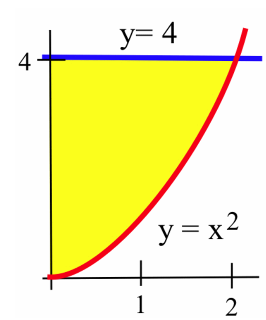


55. Given that $A'(x) = \tan(x)$, find $\mathbf{D}(A(3x))$, $\mathbf{D}\left(A(x^2)\right)$ and $\mathbf{D}(A(\sin(x)))$.

56. Given that $B'(x) = \sec(x)$, find $\mathbf{D}(B(3x))$, $\mathbf{D}\left(B(x^2)\right)$ and $\mathbf{D}(B(\sin(x)))$.

In 57–68, apply Leibniz's Rule.

57. $\dfrac{d}{dx}\left[\int_1^{5x} \sqrt{1+t}\,dt\right]$

58. $\dfrac{d}{dx}\left[\int_2^{x^2} \sqrt{1+t}\,dt\right]$

59. $\dfrac{d}{dx}\left[\int_0^{\sin(x)} \sqrt{1+t}\,dt\right]$

60. $\dfrac{d}{dx}\left[\int_1^{2+3x} \left(t^2+5\right)\,dt\right]$

61. $\dfrac{d}{dx}\left[\int_0^{1-2x} \left(3t^2+2\right)\,dt\right]$

62. $\dfrac{d}{dx}\left[\int_x^{9} \left(3t^2+2\right)\,dt\right]$

63. $\dfrac{d}{dx}\left[\int_x^{\pi} \cos(3t)\,dt\right]$

64. $\dfrac{d}{dx}\left[\int_{7x}^{\pi} \cos(2t)\,dt\right]$

65. $\dfrac{d}{dx}\left[\int_x^{x^2} \tan(t)\,dt\right]$

66. $\dfrac{d}{dx}\left[\int_0^{\pi} \cos(3t)\,dt\right]$

67. $\dfrac{d}{dx}\left[\int_2^{\ln(x)} 5t\cdot\cos(3t)\,dt\right]$

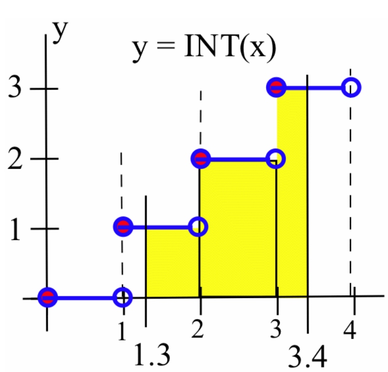
68. $\dfrac{d}{dx}\left[\int_0^{\pi} \tan(7t)\,dt\right]$

69. $\dfrac{d}{dy}\left[\int_0^{y^2} \tan(\theta)\,d\theta\right]$

4.5 Practice Answers

1. $A(1) = 1$, $A(2) = 1.5$, $A(3) = 1$, $A(4) = 0.5$; $A'(x) = fgx)$ so $A'(1) = g(1) = 1$, $A'(2) = g(2) = 0$, $A'(3) = -1$, $A'(4) = 0$.

2. $F(x) = x^3 - x$ is an antiderivative of $f(x) = 3x^2 - 1$ so:
$$\int_1^3 \left[3x^2-1\right]\,dx = \left[x^3 - x\right]_1^3 = \left[3^3 - 3\right] - \left[1^3 - 1\right] = 24$$
$F(x) = x^3 - x + 7$ is another antiderivative of $f(x) = 3x^2 - 1$ so:
$$\int_1^3 \left[3x^2-1\right]\,dx = \left[x^3 - x + 7\right]_1^3 = \left[3^3 - 3 + 7\right] - \left[1^3 - 1 + 7\right] = 24$$
No matter which antiderivative of $f(x) = 3x^2 - 1$ you use, the value of the definite integral $\int_1^3 \left[3x^2-1\right]\,dx$ is 24.



3. Because $f(x) = \lfloor x \rfloor$ is not continuous on $[1.3, 3.4]$ we cannot use the Fundamental Theorem of Calculus. Instead, we can think of the definite integral as an area (see margin figure) and compute:
$$\int_{1.3}^{3.4} \lfloor x \rfloor\,dx = 3.9$$

4. First break the solid into "slices" and approximate the volume of the k-th slice by $(3-c_k)^2 \cdot \Delta x_k$ where c_k is any point in the k-th subinterval. Next add up these approximate volumes to get a Riemann Sum:
$$\sum_{k=1}^{n} (3-c_k)^2 \cdot \Delta x_k$$

and then take the limit of these Riemann sums as the mesh of the partitions approaches 0 (and $n \to \infty$, where n is the number of subintervals in the partition):

$$\begin{aligned}
\lim_{\|\mathcal{P}\| \to 0} \left[\sum_{k=1}^{n} (3 - c_k)^2 \cdot \Delta x_k \right] &= \int_0^2 (3 - x)^2 \, dx \\
&= \int_0^2 \left(9 - 6x + x^2\right) dx \\
&= \left[9x - 3x^2 + \frac{1}{3}x^3 \right]_0^2 \\
&= \left[18 - 12 + \frac{8}{3} \right] - [0 - 0 + 0] = \frac{26}{3}
\end{aligned}$$

5. $\dfrac{d}{dx}\left[\displaystyle\int_0^{x^3} \sin(t)\,dt\right] = \sin(x^3) \cdot \dfrac{d}{dx}\left[x^3\right] = 3x^2 \sin(x^3)$

4.6 Finding Antiderivatives

In order to use the second part of the Fundamental Theorem of Calculus, we need an antiderivative of the integrand, but sometimes it is not easy to find one. This section collects some of the information we already know about general properties of antiderivatives and about antiderivatives of particular functions. It shows how to use this information to find antiderivatives of more complicated functions and introduces a "change of variable" technique to make that job easier.

Indefinite Integrals and Antiderivatives

Antiderivatives arise so often that there is a special notation to indicate the antiderivative of a function:

If you've been wondering why we called $\int_a^b f(t)\,dt$ a **definite** integral, now you know. A definite integral has specific upper and lower limits, while an indefinite integral does not.

> $\int f(x)\,dx$, read as "the **indefinite integral** of f" or as "the antiderivatives of f," represents the collection (or family) of all functions whose derivatives are f.

If F is an antiderivative of f, then any member of the family $\int f(x)\,dx$ has the form $F(x) + C$ for some constant C. We write $\int f(x)\,dx = F(x) + C$, where C represents an arbitrary constant. There are no small families in the world of antiderivatives: if f has one antiderivative F, then f has an *infinite* number of antiderivatives and each has the form $F(x) + C$, which means there are many ways to write a particular indefinite integral and some of them may look very different. You can check that $F(x) = \sin^2(x)$, $G(x) = -\cos^2(x)$ and $H(x) = 2\sin^2(x) + \cos^2(x)$ all have the same derivative, $f(x) = 2\sin(x)\cos(x)$, so the indefinite integral of $2\sin(x)\cos(x)$, $\int 2\sin(x)\cos(x)\,dx$, can be written in several ways: $\sin^2(x) + C$ or $-\cos^2(x) + K$ or $2\sin^2(x) + \cos^2(x) + C$.

Practice 1. Verify that $\int 2\tan(x)\cdot\sec^2(x)\,dx = \tan^2(x) + C$ and that $\int 2\tan(x)\cdot\sec^2(x)\,dx = \sec^2(x) + K$.

Properties of Antiderivatives (Indefinite Integrals)

These sum, difference and constant-multiple properties follow directly from corresponding properties for derivatives.

> If f and g are integrable functions, then
>
> - $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
> - $\int k\cdot f(x)\,dx = k\cdot\int f(x)\,dx$

Although we know general rules for *derivatives* of products and quotients, unfortunately there are no easy general patterns for *antiderivatives* of products and quotients — we will only be able to add one more general property to this list (in Section 8.2).

We already know antiderivatives for several important functions.

Constant Functions: $\int k\,dx = kx + C$

Powers of x: $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$ if $p \neq -1$, $\int \frac{1}{x}\,dx = \ln|x| + C$

Exponential Functions: $\int e^x\,dx = e^x + C$

Trig Functions: $\int \cos(x)\,dx = \sin(x) + C$, $\int \sin(x)\,dx = -\cos(x) + C$

$\int \sec^2(x)\,dx = \tan(x) + C$, $\int \csc^2(x)\,dx = -\cot(x) + C$

$\int \sec(x)\cdot\tan(x)\,dx = \sec(x) + C$, $\int \csc(x)\cdot\cot(x)\,dx = -\csc(x) + C$

Our list of antiderivatives of particular functions will grow in coming chapters and will eventually include antiderivatives of additional trigonometric functions, the inverse trigonometric functions, logarithms, rational functions and more. (See Appendix I.)

All of these antiderivatives can be verified by differentiating. For $\int \frac{1}{x}\,dx$ you may be wondering about the presence of the absolute value signs in the antiderivative. If $x > 0$, you can check that:

$$\mathbf{D}(\ln(|x|)) = \mathbf{D}(\ln(x)) = \frac{1}{x}$$

If $x < 0$, then you can check that:

$$\mathbf{D}(\ln(|x|)) = \mathbf{D}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$$

When computing a definite integral of the form $\int_a^b \frac{1}{x}\,dx$, either a and b will both be positive or both be negative, because the integrand is not defined at $x = 0$, so $x = 0$ cannot be included in the interval of integration.

Antiderivatives of More Complicated Functions

Antiderivatives are very sensitive to small changes in the integrand, so we should be very careful.

Fortunately, an antiderivative can always be checked by differentiating, so even though we may not find the correct antiderivative, we should be able to determine whether or not an antiderivative candidate is actually an antiderivative.

Example 1. We know $\mathbf{D}(\sin(x)) = \cos(x)$, so $\int \cos(x)\,dx = \sin(x) + C$.
Find: (a) $\int \cos(2x+3)\,dx$ (b) $\int \cos(5x-7)\,dx$ (c) $\int \cos(x^2)\,dx$

Solution. (a) Because $\sin(x)$ is an antiderivative of $\cos(x)$, it is reasonable to hope that $\sin(2x+3)$ will be an antiderivative of $\cos(2x+3)$. Unfortunately, we see that $\mathbf{D}(\sin(2x+3)) = \cos(2x+3)\cdot 2$, exactly twice the result we want. Let's try again by modifying our "guess" to be half the original guess:

$$\mathbf{D}\left(\frac{1}{2}\sin(2x+3)\right) = \frac{1}{2}\cos(2x+3)\cdot 2 = \cos(2x+3)$$

which is what we want, so $\int \cos(2x+3)\,dx = \frac{1}{2}\sin(2x+3) + C$.

(b) $\mathbf{D}(\sin(5x-7)) = \cos(5x-7)\cdot 5$, so dividing the original guess by 5 we get $\mathbf{D}\left(\frac{1}{5}\sin(5x-7)\right) = \frac{1}{5}\cos(5x-7)\cdot 5 = \cos(5x-7)$ and conclude that $\int \cos(5x-7)\,dx = \frac{1}{5}\sin(5x-7) + C$.

(c) $\mathbf{D}\left(\sin(x^2)\right) = \cos(x^2)\cdot 2x$. It was easy enough in parts (a) and (b) to modify our "guesses" to eliminate the constants 2 and 5, but here the x is much harder to eliminate:

$$\begin{aligned}\mathbf{D}\left(\frac{1}{2x}\sin\left(x^2\right)\right) &= \mathbf{D}\left(\frac{\sin(x^2)}{2x}\right)\\ &= \frac{2x\cdot\mathbf{D}\left(\sin(x^2)\right)-\sin(x^2)\cdot\mathbf{D}(2x)}{(2x)^2)}\\ &= \frac{(2x)^2\cos(x^2)-2\sin(x^2)}{(2x)^2}\\ &= \cos(x^2)-\frac{\sin(x^2)}{2x^2}\neq\cos(x^2)\end{aligned}$$

Our guess did not check out—we're stuck. ◀

Advanced mathematical techniques beyond the scope of this text can show that $\cos(x^2)$ does not have an "elementary" antiderivative composed of polynomials, roots, trigonometric functions, exponential functions or their inverses.

The value of a definite integral of $\cos(x^2)$ could still be approximated as accurately as needed by using Riemann sums or one of the numerical techniques in Sections 4.9 and 8.7, but no matter how hard we try, we cannot find a concise formula for an antiderivative of $\cos(x^2)$ in order to use the Fundamental Theorem of Calculus. Even a simple-looking integrand can be very difficult. At this point, there is no quick way to tell the difference between an "easy" indefinite integral and a "difficult" or "impossible" one.

Getting the Constants Right

The previous example illustrated one technique for finding antiderivatives: "guess" the form of the answer, differentiate your "guess" and then modify your original "guess" so its derivative is exactly what you want it to be.

Example 2. Knowing that $\int \sec^2(x)\,dx = \tan(x)+C$ and $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}+C$, find (a) $\int \sec^2(3x+7)\,dx$ (b) $\int \frac{1}{\sqrt{5x+3}}\,dx$.

Solution. (a) If we "guess" an answer of $\tan(3x+7)$ and then differentiate it, we get $\mathbf{D}\left(\tan(3x+7)\right) = \sec^2(3x+7)\cdot\mathbf{D}(3x+7) = 3\sec^2(3x+7)$, which is three times what we want. If we divide our original guess by 3 and try again, we have:

$$\begin{aligned}\mathbf{D}\left(\frac{1}{3}\tan(3x+7)\right) &= \frac{1}{3}\mathbf{D}\left(\tan(3x+7)\right) = \frac{1}{3}\sec^2(3x+7)\cdot 3\\ &= \sec^2(3x+7)\end{aligned}$$

so $\int \sec^2(3x+7)\,dx = \frac{1}{3}\tan(3x+7)+C$.

(b) If we "guess" $2\sqrt{5x+3}$ and then differentiate it, we get:

$$\mathbf{D}\left(2\left(5x+3\right)^{\frac{1}{2}}\right) = 2\cdot\frac{1}{2}\left(5x+3\right)^{-\frac{1}{2}}\mathbf{D}(5x+3) = 5\cdot(5x+3)^{-\frac{1}{2}}$$

which is five times what we want. Dividing our guess by 5 and differentiating, we have:

$$\mathbf{D}\left(\frac{2}{5}(5x+3)^{\frac{1}{2}}\right)=\frac{2}{5}\cdot\frac{1}{2}(5x+3)^{-\frac{1}{2}}\cdot 5=\frac{1}{\sqrt{5x+3}}$$

so $\displaystyle\int\frac{1}{\sqrt{5x+3}}\,dx=\frac{2}{5}\sqrt{5x+3}+C.$ ◀

Practice 2. Find $\displaystyle\int \sec^2(7x)\,dx$ and $\displaystyle\int\frac{1}{\sqrt{3x+8}}\,dx$.

The "guess and check" method is a very effective technique if you can make a good first guess, one that misses the desired result only by a constant multiple. In that situation, just divide the first guess by the unwanted constant multiple. If the derivative of your guess misses by something other than a constant multiple, then more drastic modifications are needed. Sometimes the next technique can help.

Making Patterns More Obvious: Changing the Variable

Successful integration is mostly a matter of recognizing patterns. The "change of variable" technique can make some underlying patterns of an integral easier to recognize. Essentially, the technique involves rewriting an integral that is originally in terms of one variable, say x, in terms of another variable, say u, with the hope that it will be easier to find an antiderivative of the new integrand.

For example, we can rewrite $\displaystyle\int\cos(5x+1)\,dx$ by setting $u=5x+1$. Then $\cos(5x+1)$ becomes $\cos(u)$ but we must also convert the dx in the original integral. We know that $\dfrac{du}{dx}=5$, so rewriting this last expression in differential notation, we get $du=5\,dx$; isolating dx yields $dx=\frac{1}{5}\,du$ so:

We first discussed differential notation in Section 2.8; although you may not have used them much in differential calculus, you will now use them extensively.

$$\int\cos(5x+1)\,dx=\int\cos(u)\cdot\frac{1}{5}\,du=\frac{1}{5}\int\cos(u)\,du$$

This new integral is easier:

$$\frac{1}{5}\int\cos(u)\,du=\frac{1}{5}\sin(u)+C$$

but our original problem was in terms of x and our answer is in terms of u, so we must "resubstitute" using the relationship $u=5x+1$:

$$\frac{1}{5}\sin(u)+C=\frac{1}{5}\sin(5x+1)+C$$

We can now conclude that:

$$\int\cos(5x+1)\,dx=\frac{1}{5}\sin(5x+1)+C$$

As always, you can check this result by differentiating.

We can summarize the steps of this "change of variable" (or "u-substitution") method as:

Often u is set equal to some "interior" part of the original integrand function.

- set a new variable, say u, equal to some function of the original variable x
- calculate the differential du in terms of x and dx
- rewrite the original integral in terms of u and du
- integrate the new integral to get an answer in terms of u
- resubstitute for u to get a result in terms of the original variable x

Example 3. Make the suggested change of variable, rewrite each integral in terms of u and du, and evaluate the integral.

(a) $\int \cos(x) \cdot e^{\sin(x)}\, dx \quad \text{with} \quad u = \sin(x)$

(b) $\int \frac{2x}{5+x^2}\, dx \quad \text{with} \quad u = 5 + x^2$

Solution. (a) $u = \sin(x) \Rightarrow du = \cos(x)\, dx$ and $e^{\sin(x)} = e^u$:

$$\int \cos(x) e^{\sin(x)}\, dx = \int e^u\, du = e^u + C = e^{\sin(x)} + C$$

(b) $u = 5 + x^2 \Rightarrow du = 2x\, dx$, so:

$$\int \frac{2x}{5+x^2}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln\left|5 + x^2\right| + C$$

Because $5 + x^2 > 0$, we can also write the answer as $\ln\left(5 + x^2\right)$. ◀

In each example, the change of variable did not find the antiderivative, but it did make the pattern of the integrand more obvious, which in turn made it easier to determine an antiderivative.

Practice 3. Make the suggested change of variable, rewrite each integral in terms of u and du and evaluate the integral.

(a) $\int (7x+5)^3\, dx \quad \text{with} \quad u = 7x + 5$

(b) $\int 3x^2 \cdot \sin\left(x^3 - 1\right)\, dx \quad \text{with} \quad u = x^3 - 1$

The previous examples have supplied a suggested substitution, but in the future *you* will need to decide what u should equal. Unfortunately there are no rules that guarantee your choice will lead to an easier integral—sometimes you will need to resort to trial and error until you find a particular u-substitution that works for your integrand. There is, however, a "rule of thumb" that frequently results in easier integrals. Even though the following suggestion comes with no guarantees, it is often worth trying.

A "Rule of Thumb" for Changing the Variable

If part of the integrand consists of a composition of functions, $f(g(x))$, try setting $u = g(x)$, the "inner" function.

The key to becoming skilled at selecting a good u and correctly making the substitution is **practice**.

If part of the integrand is being raised to a power, try setting u equal to the part being raised to the power. For example, if the integrand includes $(3+\sin(x))^5$, try $u = 3+\sin(x)$. If part of the integrand involves a trigonometric (or exponential or logarithmic) function of another function, try setting u equal to the "inside" function: if the integrand includes the function $\sin\left(3+x^2\right)$, try $u = 3+x^2$.

Example 4. Select a u for each integrand and rewrite the associated integral in terms of u and du.

(a) $\displaystyle\int \cos(3x)\sqrt{2+\sin(3x)}\,dx$ (b) $\displaystyle\int \frac{5e^x}{2+e^x}\,dx$ (c) $\displaystyle\int e^x \cdot \sin\left(e^x\right)\,dx$

Solution. (a) If $u = 2+\sin(3x)$, $du = 3\cos(3x)\,dx \Rightarrow \frac{1}{3}\,du = \cos(3x)\,dx$ so the integral becomes $\displaystyle\int \frac{1}{3}\sqrt{u}\,du$. (b) With $u = 2+e^x \Rightarrow du = e^x\,dx$, the integral becomes $\displaystyle\int \frac{5}{u}\,du$. (c) With $u = e^x \Rightarrow du = e^x\,dx$, the integral becomes $\displaystyle\int \sin(u)\,du$. ◀

Changing Variables with Definite Integrals

If we need to change variables in a *definite* integral, we have two choices:

- First work out the corresponding *indefinite* integral and then use that antiderivative and FTC[2] to evaluate the definite integral.
- Change variables in the definite integral, which requires changing the limits of integration from x limits to u limits.

For the second option, if the original integral had endpoints $x = a$ and $x = b$, and we make the substitution $u = g(x) \Rightarrow du = g'(x)\,dx$, then the new integral will have endpoints $u = g(a)$ and $u = g(b)$:

$$\int_{x=a}^{x=b} f(g(x)) \cdot g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

Example 5. Evaluate $\displaystyle\int_0^1 (3x-1)^4\,dx$.

Solution. Using the first option with $u = 3x-1 \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$, the corresponding indefinite integral becomes:

$$\int (3x-1)^4\,dx = \int u^4 \cdot \frac{1}{3}\,du = \frac{1}{3}\cdot\frac{1}{5}u^5 + C = \frac{1}{15}(3x-1)^5 + C$$

We now use this result to evaluate the original definite integral:

$$\int_0^1 (3x-1)^4\,dx = \left[\frac{1}{15}(3x-1)^5\right]_0^1 = \left[\frac{1}{15}\cdot 2^5\right] - \left[\frac{1}{15}\cdot(-1)^5\right]$$
$$= \frac{32}{15} - \frac{-1}{15} = \frac{33}{15} = \frac{11}{5}$$

For the second option, we make the same substitution $u = 3x-1 \Rightarrow \frac{1}{3}\,du = dx$ while also computing $x = 0 \Rightarrow u = 3\cdot 0 - 1 = -1$ and $x = 1 \Rightarrow u = 3\cdot 1 - 1 = 2$:

Both options require roughly the same amount of work and computation. In practice you should choose the option that seems easiest for you and poses the least risk of error.

$$\int_{x=0}^{x=1} (3x-1)^4\,dx = \int_{u=-1}^{u=2} \frac{1}{3}u^4\,du = \frac{1}{3}\cdot\frac{1}{5}u^5\bigg|_{-1}^{2} = \frac{2^5}{15} - \frac{(-1)^5}{15} = \frac{33}{15}$$

We arrive at the same answer either way. ◀

Practice 4. If the original integrals in Example 4 had endpoints (a) $x = 0$ to $x = \pi$ (b) $x = 0$ to $x = 2$ or (c) $x = 0$ to $x = \ln(3)$, then the new integrals should have what endpoints?

Special Transformations for $\int \sin^2(x)\,dx$ and $\int \cos^2(x)\,dx$

The integrals of $\sin^2(x)$ and $\cos^2(x)$ arise often, and we can find their antiderivatives with the help of some trigonometric identities. Solving the first identity in the margin for $\sin^2(x)$, we get:

$\cos(2x) = 1 - 2\sin^2(x)$
$\cos(2x) = 2\cos^2(x) - 1$
$\sin(2x) = 2\sin(x)\cos(x)$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

and solving the second identity for $\cos^2(x)$, we get:

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

Integrating the first of these new identities yields:

$$\int \sin^2(x)\,dx = \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

Using the identity $\sin(2x) = 2\sin(x)\cos(x)$, we can also write:

$$\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + C$$

In practice, it's easier to remember the new trig identities and use them to work out these antiderivatives, rather than memorizing the antiderivatives directly.

Similarly, using $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ yields:

$$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C = \frac{1}{2}x + \frac{1}{2}\sin(x)\cos(x) + C$$

4.6 Problems

For Problems 1–4, put $f(x) = x^2$ and $g(x) = x$ to verify the inequality.

1. $\int_1^2 f(x) \cdot g(x)\,dx \neq \left(\int_1^2 f(x)\,dx\right)\left(\int_1^2 g(x)\,dx\right)$
2. $\int_1^2 \frac{f(x)}{g(x)}\,dx \neq \frac{\int_1^2 f(x)\,dx}{\int_1^2 g(x)\,dx}$
3. $\int_0^1 f(x) \cdot g(x)\,dx \neq \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right)$
4. $\int_1^4 \frac{f(x)}{g(x)}\,dx \neq \frac{\int_1^4 f(x)\,dx}{\int_1^4 g(x)\,dx}$

For 5–14, use the suggested u to find du and rewrite the integral in terms of u and du. Then find an antiderivative in terms of u and, finally, rewrite your answer in terms of x.

5. $\int \cos(3x)\,dx, \quad u = 3x$
6. $\int \sin(7x)\,dx, \quad u = 7x$
7. $\int e^x \sin(2+e^x)\,dx, \quad u = 2 + e^x$
8. $\int e^{5x}\,dx, \quad u = 5x$
9. $\int \cos(x)\sec^2(\sin(x))\,dx, \quad u = \sin(x)$
10. $\int \frac{\cos(x)}{\sin(x)}\,dx, \quad u = \sin(x)$
11. $\int \frac{5}{3+2x}\,dx, \quad u = 3 + 2x$
12. $\int x^2\left(5+x^3\right)^7\,dx, \quad u = 5 + x^3$
13. $\int x^2 \sin\left(1+x^3\right)\,dx, \quad u = 1 + x^3$
14. $\int \frac{e^x}{1+e^x}\,dx, \quad u = 1 + e^x$

For 15–26, use the change-of-variable technique to evaluate the indefinite integral.

15. $\int \cos(4x)\,dx$
16. $\int e^{3x}\,dx$
17. $\int x^3\left(5+x^4\right)^{11}\,dx$
18. $\int x \cdot \sin\left(x^2\right)\,dx$
19. $\int \frac{3x^2}{2+x^3}\,dx$
20. $\int \frac{\sin(x)}{\cos(x)}\,dx$
21. $\int \frac{\ln(x)}{x}\,dx$
22. $\int x\sqrt{1+x^2}\,dx$
23. $\int (1+3x)^7\,dx$
24. $\int \frac{1}{x} \cdot \sin(\ln(x))\,dx$
25. $\int e^x \cdot \sec(e^x) \cdot \tan(e^x)\,dx$
26. $\int \frac{1}{\sqrt{x}} \cos\left(\sqrt{x}\right)\,dx$

In 27–42, evaluate the integral.

27. $\int_0^{\frac{\pi}{2}} \cos(3x)\,dx$
28. $\int_0^{\pi} \cos(4x)\,dx$
29. $\int_0^1 e^x \cdot \sin(2+e^x)\,dx$
30. $\int_0^1 e^{5x}\,dx$
31. $\int_{-1}^1 x^2\left(1+x^3\right)^5\,dx$
32. $\int_0^1 x^4\left(x^5-1\right)^{10}\,dx$
33. $\int_0^2 \frac{5}{3+2x}\,dx$
34. $\int_0^{\ln(3)} \frac{e^x}{1+e^x}\,dx$
35. $\int_0^1 x\sqrt{1-x^2}\,dx$
36. $\int_2^5 \frac{2}{1+x}\,dx$
37. $\int_0^1 \sqrt{1+3x}\,dx$
38. $\int_0^1 \frac{1}{\sqrt{1+3x}}\,dx$
39. $\int \sin^2(5x)\,dx$
40. $\int \cos^2(3x)\,dx$
41. $\int \left[\frac{1}{2} - \sin^2(x)\right]\,dx$
42. $\int \left[e^x + \sin^2(x)\right]\,dx$
43. Find the area under one arch of the graph of $y = \sin^2(x)$.
44. Evaluate $\int_0^{2\pi} \sin^2(x)\,dx$.

In 45–53, expand the integrand and then find an antiderivative.

45. $\int \left(x^2+1\right)^3\,dx$
46. $\int \left(x^3+5\right)^2\,dx$
47. $\int (e^x+1)^2\,dx$
48. $\int \left(x^2+3x-2\right)^2\,dx$
49. $\int (x^2+1)(x^3+5)\,dx$
50. $\int (7+\sin(x))^2\,dx$
51. $\int e^x\left(e^x+e^{3x}\right)\,dx$
52. $\int (2+\sin(x))\sin(x)\,dx$
53. $\int \sqrt{x}\left(x^2+3x-2\right)\,dx$

In 54–64, divide, then find an antiderivative.

54. $\int \frac{x+1}{x}\,dx$

55. $\int \frac{3x}{x+1}\,dx$

56. $\int \frac{x-1}{x+2}\,dx$

57. $\int \frac{x^2-1}{x+1}\,dx$

58. $\int \frac{2x^2-13x+15}{x-1}\,dx$

59. $\int \frac{2x^2-13x+18}{x-1}\,dx$

60. $\int \frac{2x^2-13x+11}{x-1}\,dx$

61. $\int \frac{x+2}{x-1}\,dx$

62. $\int \frac{e^x+e^{3x}}{e^x}\,dx$

63. $\int \frac{x+4}{\sqrt{x}}\,dx$

64. $\int \frac{\sqrt{x}+3}{x}\,dx$

The definite integrals in 65–70 involve areas associated with parts of circles; use your knowledge of circles and their areas to evaluate them. (Suggestion: Sketch a graph of the integrand function.)

65. $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

66. $\int_{0}^{1} \sqrt{1-x^2}\,dx$

67. $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

68. $\int_{-4}^{0} \sqrt{16-x^2}\,dx$

69. $\int_{-1}^{1} \left[2+\sqrt{1-x^2}\right]\,dx$

70. $\int_{0}^{2} \left[3-\sqrt{1-x^2}\right]\,dx$

4.6 Practice Answers

1. $\mathbf{D}\left(\tan^2(x)+C\right) = 2\tan^1(x)\cdot\mathbf{D}\left(\tan(x)\right) = 2\tan(x)\sec^2(x)$
   $\mathbf{D}\left(\sec^2(x)+C\right) = 2\sec^1(x)\cdot\mathbf{D}\left(\sec(x)\right) = 2\sec(x)\cdot\sec(x)\tan(x)$

2. We know $\mathbf{D}\left(\tan(x)\right) = \sec^2(x)$, so it is reasonable to try $\tan(7x)$: $\mathbf{D}\left(\tan(7x)\right) = \sec^2(7x)\cdot\mathbf{D}(7x) = 7\sec^2(7x)$, a result seven times the result we want, so divide the original "guess" by 7 and try again:
   $$\mathbf{D}\left(\frac{1}{7}\tan(7x)\right) = \frac{1}{7}\sec^2(7x)\cdot 7 = \sec^2(7x)$$
   so $\int \sec^2(7x)\,dx = \frac{1}{7}\tan(7x)+C$.
   $\mathbf{D}\left((3x+8)^{\frac{1}{2}}\right) = \frac{1}{2}(3x+8)^{-\frac{1}{2}}\,\mathbf{D}(3x+8) = \frac{3}{2}(3x+8)^{-\frac{1}{2}}$ so multiply our original "guess" by $\frac{2}{3}$:
   $$\mathbf{D}\left(\frac{2}{3}(3x+8)^{\frac{1}{2}}\right) = \frac{2}{3}\cdot\frac{1}{2}\cdot(3x+8)^{-\frac{1}{2}}\cdot\mathbf{D}(3x+8) = \frac{2}{3}\cdot\frac{3}{2}\cdot\frac{1}{\sqrt{3x+8}}$$
   hence $\int \frac{1}{\sqrt{3x+8}}\,dx = \frac{2}{3}\sqrt{3x+8}+C$.

3. (a) $u = 7x+5 \Rightarrow du = 7\,dx \Rightarrow dx = \frac{1}{7}\,du$ so:
   $$\int (7x+5)^3\;dx = \int u^3\cdot\frac{1}{7}\,du = \frac{1}{7}\cdot\frac{1}{4}u^4 + C = \frac{1}{28}\left(7x+5\right)^4 + C$$
   (b) $u = x^3-1 \Rightarrow du = 3x^2\,dx$ so $\int \sin(x^3-1)\cdot 3x^2\,dx$ becomes:
   $$\int \sin(u)\,du = -\cos(u)+C = -\cos\left(x^3-1\right)+C$$

4. (a) $u = 2+\sin(3x)$ so $x = 0 \Rightarrow u = 2+\sin(3\cdot 0) = 2$ and $x = \pi \Rightarrow u = 2+\sin(3\pi) = 2$. (This integral is now easy; why?)
   (b) $u = 2+e^x$ so $x = 0 \Rightarrow u = 2+e^0 = 3$ and $x = 2 \Rightarrow u = 2+e^2$
   (c) $u = e^x$ so $x = 0 \Rightarrow u = e^0 = 1$ and $x = \ln(3) \Rightarrow u = e^{\ln(3)} = 3$

4.7 First Applications of Definite Integrals

The development of calculus by Newton and Leibniz was a vital step in the advancement of pure mathematics, but Newton also advanced the sciences and applied mathematics. Not only did he discover theoretical results, he immediately used those results to answer important questions about gravity and motion. The success of these applications of mathematics to the physical sciences helped establish what we now take for granted: mathematics can and should be used to answer questions about the world.

Newton applied mathematics to the outstanding problems of his day, problems primarily in the field of physics. During the intervening 300-plus years, thousands upon thousands of people have continued these theoretical and applied traditions, using mathematics to help develop our understanding of the physical and biological sciences, as well as the behavioral sciences and economics. Mathematics is still used to answer new questions in physics and engineering, but it is also important for modeling ecological processes, for understanding the behavior of DNA, for determining how the brain works, and even for devising financial strategies. The mathematics you are learning now can help you become part of this tradition, and you might even use it to add to our understanding of the world.

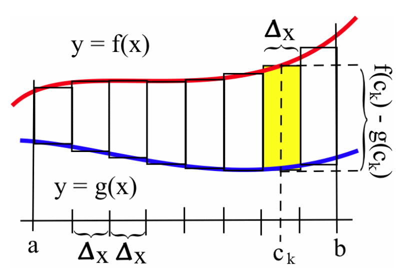
It is important to understand the special applications of integration we will study in case you need to use those particular applications. But it is also important that you understand the *process* of building models with integrals so you can apply that process to other situations in a variety of fields of study. Conceptually, converting an applied problem to a Riemann sum is the most valuable step.

Typically, it is also the most challenging.

Area between Two Curves

We have already used integrals to find the area between the graph of a function and the horizontal axis. We can also use integrals to find the area between the graphs of two functions.

If $f(x) \geq g(x)$ for all x in $[a, b]$, then we can approximate the area between the graphs of f and g by partitioning the interval $[a, b]$ and forming a Riemann sum (see margin). The height of each rectangle is $f(c_k) - g(c_k)$ so the area of the k-th rectangle is:



$$(\text{height}) \cdot (\text{base}) = [f(c_k) - g(c_k)] \cdot \Delta x_k$$

and an approximation of the total area is given by

$$\sum_{k=1}^{n} [f(c_k) - g(c_k)] \cdot \Delta x_k$$

which is a Riemann sum.

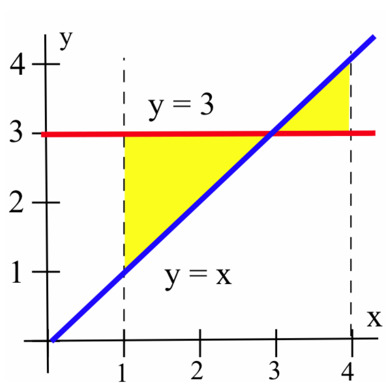
The limit of this Riemann sum, as the mesh of the partitions approaches 0, is a definite integral:

$$\int_a^b [f(x) - g(x)]\, dx$$

We will sometimes use an arrow to indicate "the limit of the Riemann sum as the mesh of the partitions approaches 0," writing:

$$\sum_{k=1}^{n} [f(c_k) - g(c_k)] \cdot \Delta x_k \longrightarrow \int_a^b [f(x) - g(x)]\, dx$$

If	$T(x) \geq B(x)$ for $a \leq x \leq b$
then	the area of the region bounded by the graphs of the "top" function $T(x)$, the "bottom" function $B(x)$, and the lines $x = a$ and $x = b$ is given by: $\int_a^b [T(x) - B(x)]\, dx$



Example 1. Find the area bounded between the graphs of $f(x) = x$ and $g(x) = 3$ for $1 \leq x \leq 4$.

Solution. It is clear from the margin figure that the area between f and g is 2.5 square units. Using the integration procedure above, we need to identify a "top" function and a "bottom" function. For $1 \leq x \leq 3$, $g(x) = 3 \geq x = f(x)$ so the area of the left-hand triangle is given by the integral:

$$\int_1^3 [3 - x]\, dx = \left[3x - \frac{1}{2}x^2\right]_1^3 = \left[9 - \frac{9}{2}\right] - \left[3 - \frac{1}{2}\right] = 2$$

For the interval $3 \leq x \leq 4$, $g(x) = 3 \leq x = f(x)$ so the area of the right-hand triangle is given by the integral:

$$\int_3^4 [x - 3]\, dx = \left[\frac{1}{2}x^2 - 3x\right]_3^4 = [8 - 12] - \left[\frac{9}{2} - 9\right] = \frac{1}{2}$$

Adding these two areas, we get $2 + 0.5 = 2.5$. ◀

If we had mindlessly integrated in the previous Example without consulting a graph:

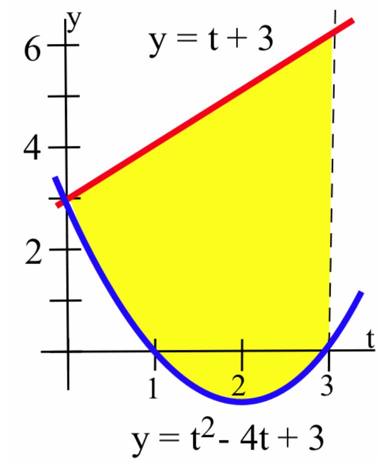
$$\int_1^4 [3 - x]\, dx = \left[3x - \frac{1}{2}x^2\right]_1^4 = [12 - 8] - \left[3 - \frac{1}{2}\right] = \frac{3}{2}$$

we would have arrived at an incorrect answer.

Graphing the region in question to determine which function is on "top" and which is on "bottom" is often crucial to getting the right answer to a problem involving the area between two curves.

Practice 1. Use integrals and the graphs of $f(x) = 1 + x$ and $g(x) = 3 - x$ to determine the area between the graphs of f and g for $0 \leq x \leq 3$.

Example 2. Objects A and B start from the same location at the same time and travel along the same path with respective velocities $v_A(t) = t+3$ and $v_B(t) = t^2 - 4t + 3$ meters per second (see margin). How far ahead is A after 3 seconds? After 5 seconds?



Solution. From the graph, it appears that $v_A(t) \geq v_B(t)$, at least for $0 \leq t \leq 3$, but for the second question we need to know whether this holds for $3 \leq t \leq 5$ as well. Setting $v_A(t) = v_B(t)$ to see where the graphs intersect:

$$t+3 = t^2 - 4t + 3 \quad \Rightarrow \quad t^2 - 5t = 0 \quad \Rightarrow \quad t = 0 \text{ or } t = 5$$

Checking that $v_A(1) = 4 > 0 = v_B(1)$ (or referring to the graph), we can conclude that $v_A(t) \geq v_B(t)$ on the interval $[0,5]$.

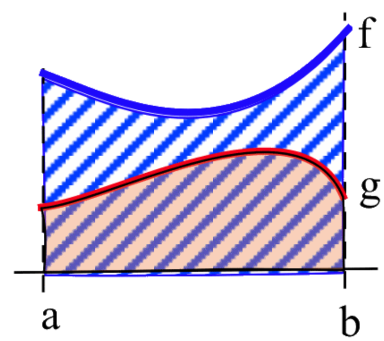
Because $v_A(t) \geq v_B(t)$, the "area" between the graphs of v_A and v_B over an interval $[0,x]$ represents the distance between the objects after x seconds. After three seconds, the distance apart is:

$$\int_0^3 [v_A(t) - v_B(t)]\, dt = \int_0^3 \left[(t+3) - (t^2 - 4t + 3)\right] dt = \int_0^3 \left[5t - t^2\right] dt$$
$$= \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^3 = \left[\frac{45}{2} - 9\right] - [0-0] = \frac{27}{2}$$

or 13.5 meters. After five seconds, the distance apart is

$$\int_0^5 [v_A(t) - v_B(t)]\, dt = \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^5 = \frac{125}{6}$$

or approximately 20.83 meters. ◀

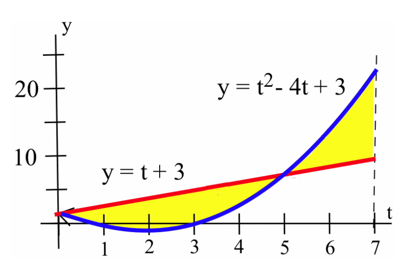


If $f(x) \geq g(x) \geq 0$ on an interval $[a,b]$ (as illustrated in the margin figure), we could have used a simpler geometric argument that the area between the graphs of f and g is just the area below the graph of f **minus** the area below the graph of g:

$$\int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx$$

which agrees with our previous result. We took a different approach at the beginning of this section, however, because it provides a nice (yet simple) example of translating a geometric or physical problem into a Riemann sum and then into a definite integral.

Example 3. Find the area of the shaded region in the margin figure.



Solution. These are the same two functions from our previous Example; in our previous solution we observed that $t+3 \geq t^2 - 4t + 3$ for $0 \leq t \leq 5$, and it is straightforward to check that $t + 3 \leq t^2 - 4t + 3$ for $t \geq 5$ (and, in particular, for $5 \leq t \leq 7$).

We therefore need to split our problem into two pieces and subtract the "bottom" function from the "top" function on each interval. The area of the left region is:

$$\int_0^5 \Big[(t+3) - (t^2 - 4t + 3)\Big]\, dt = \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^5 = \frac{125}{6}$$

(as worked out in the previous example), while the area of the region on the right is:

$$\int_5^7 \Big[(t^2 - 4t + 3) - (t+3)\Big]\, dt = \left[\frac{1}{3}t^3 - \frac{5}{2}t^2\right]_5^7 = \frac{38}{3}$$

so the total area is $\frac{125}{6} + \frac{38}{3} = \frac{67}{2} = 33.5$. ◀

Average Value of a Function

We compute the **average** (or **mean value**) of n numbers, $a_1, a_2, \ldots, a_n$ by adding them up and dividing by n:

$$\text{average} = \bar{a} = \frac{1}{n}\sum_{k=1}^{n} a_k$$

A "bar" above a quantity typically indicates the **mean** of that quanitity.

but computing the average value of a *function* requires an integral.

To estimate the average value of f on the interval $[a, b]$, we can partition $[a, b]$ into n equally long subintervals of length $\Delta x = \frac{b-a}{n}$, then choose a value c_k in each subinterval, and find the average of the function values $f(c_k)$ at those n points:

$$\bar{f} = \text{average of } f \;\approx\; \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} \;=\; \sum_{k=1}^{n} f(c_k) \cdot \frac{1}{n}$$

While this last term resembles a Riemann sum, it does not have the form $\sum f(c_k) \cdot \Delta x_k$, because $\frac{1}{n} \neq \Delta x = \frac{b-a}{n}$. But multiplying and dividing by $b - a$ yields:

$$\sum_{k=1}^{n} f(c_k) \cdot \frac{1}{n} = \sum_{k=1}^{n} f(c_k) \cdot \frac{b-a}{n} \cdot \frac{1}{b-a} = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k) \cdot \frac{b-a}{n}$$

This last (Riemann) sum converges to a definite integral:

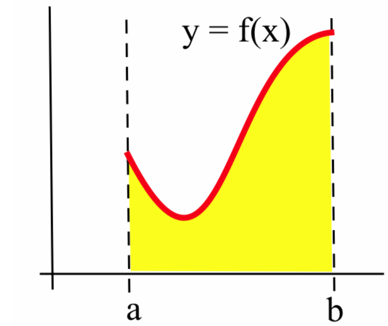
$$\frac{1}{b-a}\sum_{k=1}^{n} f(c_k) \cdot \frac{b-a}{n} = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k) \cdot \Delta x \longrightarrow \frac{1}{b-a}\int_a^b f(x)\, dx$$

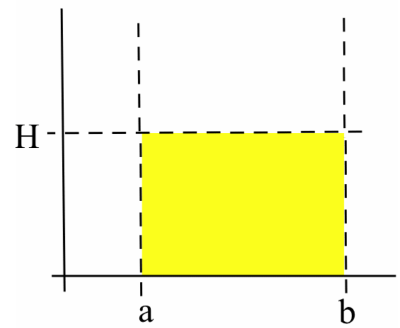
as the number of subintervals n gets larger and the mesh, $\Delta x = \frac{b-a}{n}$, approaches 0.

Definition: Average (Mean) Value of a Function

The **average value** of an integrable function f on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$





The average value of a positive function has a nice geometric interpretation. Imagine that the area under f (see margin) represents a liquid trapped above by the graph of f and on the other sides by the x-axis and the lines $x = a$ and $x = b$. If we remove the "lid" (the graph of f), the liquid would settle into the shape of a rectangle with the same area as the region under the graph of f. If the height of this rectangle is H, then the area of the rectangle is $H \cdot (b - a)$, so:

$$H \cdot (b-a) = \int_a^b f(x)\,dx \quad \Rightarrow \quad H = \frac{1}{b-a}\int_a^b f(x)\,dx$$

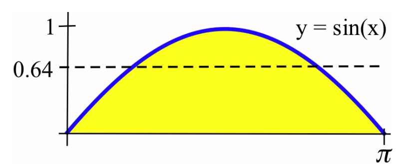
The average value of a positive function f is the height H of the rectangle whose area is the same as the area under f.

Example 4. Find the average value of $\sin(x)$ on the interval $[0, \pi]$.

Solution. Using our definition, the average value is:

$$\frac{1}{\pi - 0}\int_0^\pi \sin(x)\,dx = \frac{1}{\pi}\Big[-\cos(x)\Big]_0^\pi = \frac{1}{\pi}[(1) - (-1)] = \frac{2}{\pi} \approx 0.6366$$

A rectangle with height $\frac{2}{\pi} \approx 0.64$ on the interval $[0, \pi]$ encloses the same area as one arch of the sine curve. ◀



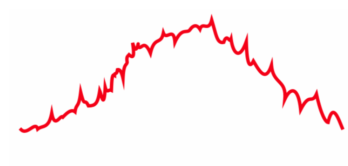
If the interval in the previous Example had been $[0, 2\pi]$, the average value would be 0. (Why?)

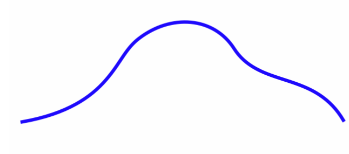
Practice 2. During a nine-hour work day, the production rate at time t hours was $r(t) = 5 + \sqrt{t}$ cars per hour. Find the average hourly production rate.

Function averages, involving means as well as more complicated techniques, are used to "smooth" data so that underlying patterns become more obvious and to remove high frequency "noise" from signals. In these situations, the value of the original function f at a point is replaced by some "average of f" over an interval including that point. If f is the graph of rather jagged data (see margin), then the 10-year average of f is the integral:

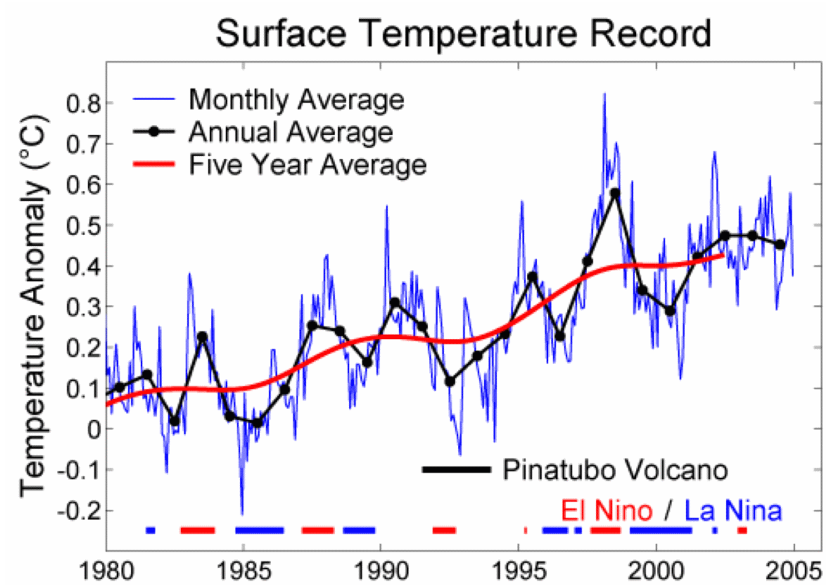
$$g(x) = \frac{1}{10}\int_{x-5}^{x+5} f(t)\,dt$$

an average of f over a timespan of five years on either side of x.


"noisy" signal


"smoothed" signal

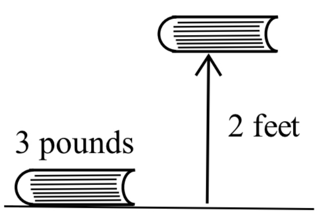
The figure below shows the graphs of a monthly average (rather "noisy" data) of surface-temperature data, an annual average (still rather "jagged") and a five-year average (a much smoother function):



This "moving average" of "noisy" data is frequently used with data such as weather information and stock prices.

Typically this "moving average" function reveals a pattern much more clearly than the original data.

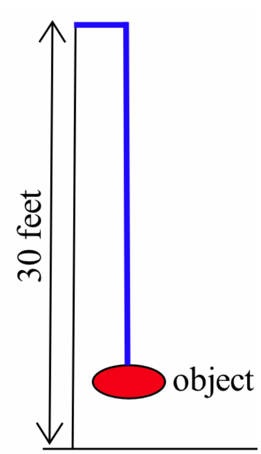
Work



In physics, the amount of work done on an object is defined as the force applied to the object times the displacement of the object (the distance the object is moved while the force is applied). Or, more succinctly:

$$\text{work} = (\text{force}) \cdot (\text{displacement})$$

If you lift a three-pound book two feet, then the force is 3 pounds (the weight of the book), and the displacement is 2 feet, so you have done $(3 \text{ pounds}) \cdot (2 \text{ feet}) = 6$ foot-pounds of work. When the applied force and the displacement are both constants, calculating work is simply a matter of multiplication.

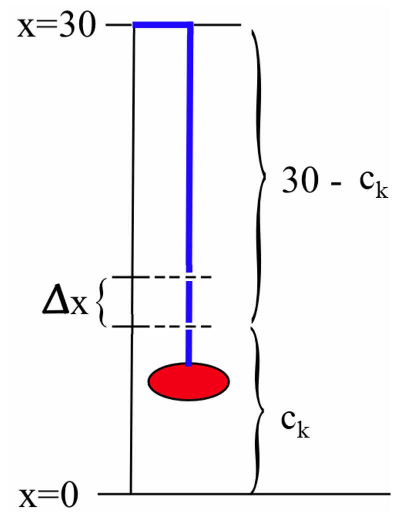


Practice 3. How much work is done lifting a 10-pound object from the ground to the top of a 30-foot building?

If either the force or the displacement varies, however, we need to use integration.

Example 5. How much work is done lifting a 10-pound object from the ground to the top of a 30-foot building using a cable that weighs 2 pounds per foot?

Solution. This is more challenging situation. We know the work needed to move the object is $(10)(30) = 300$ foot-pounds, but once we start pulling the cable onto the roof, we need to do less and less work to pull the remaining part of the cable.



Let's partition the cable into small increments so the displacement of each small piece of the cable is roughly constant. If we break the cable into n small pieces, each piece has length $\Delta x = \frac{30}{n}$, so its weight (the force required to move it) is:

$$(\Delta x \text{ ft}) \cdot \left(2\,\frac{\text{lbs}}{\text{ft}}\right) = 2\Delta x \text{ lbs}$$

If this small piece of cable is initially c_k feet above the ground, then its displacement is $30 - c_k$ feet, so the work done on this small piece is $2(30 - c_k)\Delta x$ ft-lbs and the total work done on the entire cable is (approximately):

$$\sum_{k=1}^{n} 2(30 - c_k)\Delta x \quad \longrightarrow \quad \int_0^{30} 2(30 - x)\,dx$$

Once again we have formed a Riemann sum, which converges to a definite integral as we chop the cable into smaller and smaller pieces. This integral represents the work needed to lift the cable to the roof:

$$\int_0^{30} 2(30 - x)\,dx = \int_0^{30} (60 - 2x)\,dx = 60x - x^2\Big|_0^{30}$$
$$= [1800 - 900] - [0 - 0] = 900 \text{ ft-lbs}$$

so the total work required to lift the object and the cable to the roof is $300 + 900 = 1200$ ft-lbs. ◀

Practice 4. Suppose the building in Example 5 is 50 feet tall and the cable weighs 3 pounds per foot.

(a) Compute the work done raising the object from the ground to a height of 10 feet.

(b) From a height of 10 feet to a height of 20 feet.

The situation in the previous Example and Practice problems is but one of many that arise when computing work. We will examine others in Section 5.4.

Summary

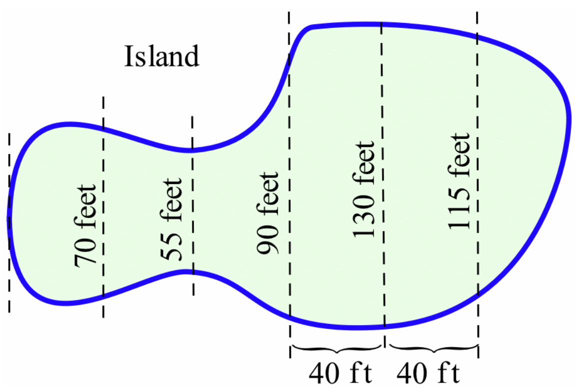
The area, average and work applications in this section merely introduce a few of the many applications of definite integrals. They illustrate the pattern of moving from an applied problem to a Riemann sum, to a definite integral and, finally, to a numerical answer. We will explore many more applications in Chapter 5.

4.7 Problems

In Problems 1–4, use the values in the table below to estimate the indicated areas.

x	f(x)	g(x)	h(x)
0	5	2	5
1	6	1	6
2	6	2	8
3	4	2	6
4	3	3	5
5	2	4	4
6	2	5	2

1. Estimate the area between f and g for $1 \le x \le 4$.
2. Estimate the area between f and g for $1 \le x \le 6$.
3. Estimate the area between f and h for $0 \le x \le 4$.
4. Estimate the area between g and h for $0 \le x \le 6$.
5. Estimate the area of the island in the figure below.



6. Estimate the area of the island in figure above if the distances between the lines is 50 feet instead of 40 feet.

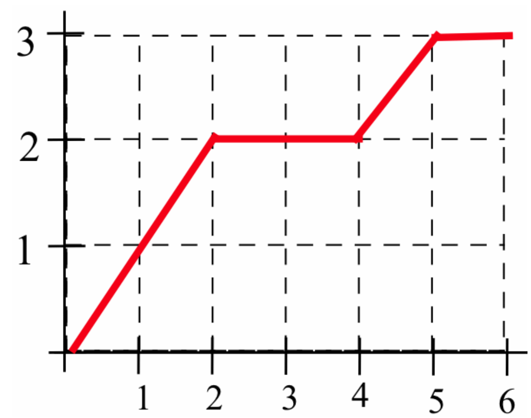
In Problems 7–18, sketch a graph of each function and find the area between the graphs of f and g for x in the given interval.

7. $f(x) = x^2 + 3$, $g(x) = 1$, $-1 \le x \le 2$
8. $f(x) = x^2 + 3$, $g(x) = 1 + x$, $0 \le x \le 3$
9. $f(x) = x^2$, $g(x) = x$, $0 \le x \le 2$
10. $f(x) = 4 - x^2$, $g(x) = x + 2$, $0 \le x \le 2$
11. $f(x) = \dfrac{1}{x}$, $g(x) = x$, $1 \le x \le e$
12. $f(x) = \sqrt{x}$, $g(x) = x$, $0 \le x \le 4$
13. $f(x) = x + 1$, $g(x) = \cos(x)$, $0 \le x \le \dfrac{\pi}{4}$
14. $f(x) = (x-1)^2$, $g(x) = x + 1$, $0 \le x \le 3$
15. $f(x) = e^x$, $g(x) = x$, $0 \le x \le 2$
16. $f(x) = \cos(x)$, $g(x) = \sin(x)$, $0 \le x \le \dfrac{\pi}{4}$
17. $f(x) = 3$, $g(x) = \sqrt{1 - x^2}$, $0 \le x \le 1$
18. $f(x) = 2$, $g(x) = \sqrt{4 - x^2}$, $-2 \le x \le 2$

In Problems 19–22, use the values of f in the table at the beginning of the page to estimate the average value of f on the indicated interval.

19. $[0.5, 4.5]$ 20. $[0.5, 6.5]$ 21. $[1.5, 3.5]$ 22. $[3.5, 6.5]$

In 23–26, find the average value of the function whose graph appears below on the given interval.

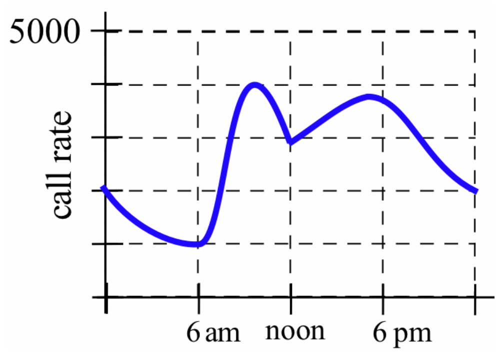


23. $[0, 2]$ 24. $[0, 4]$ 25. $[1, 6]$ 26. $[4, 6]$

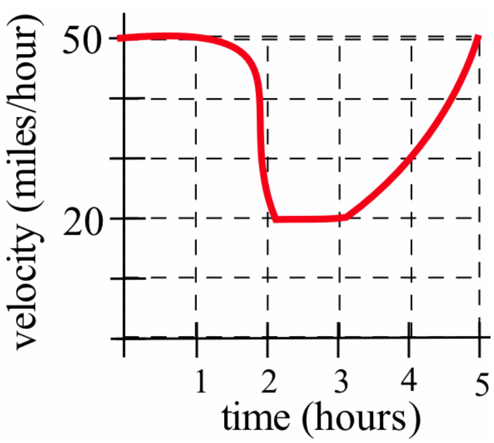
In Problems 27–32, find the average value of the given function on the indicated interval.

27. $f(x) = 2x + 1$, $0 \le x \le 4$
28. $f(x) = x^2$, $0 \le x \le 2$
29. $f(x) = x^2$, $1 \le x \le 3$
30. $f(x) = \sqrt{x}$, $0 \le x \le 4$
31. $f(x) = \sin(x)$, $0 \le x \le \pi$
32. $f(x) = \cos(x)$, $0 \le x \le \pi$
33. Calculate the average value of $f(x) = \sqrt{x}$ on the interval $[0, C]$ for $C = 1$, 9, 81, 100. What is the pattern?
34. Calculate the average value of $f(x) = x$ on the interval $[0, C]$ for $C = 1$, 10, 80, 100. What is the pattern?

35. The figure below shows the number of telephone calls per minute at a large company. Estimate the average number of calls per minute:
    (a) from 8:00 a.m. to 5:00 p.m.
    (b) from 9:00 a.m. to 1:00 p.m.



36. The figure below shows the velocity of a car during a five-hour trip.
    (a) Estimate how far the car traveled.
    (b) At what constant velocity should you drive in order to travel the same distance in five hours?



37. (a) How much work is done lifting a 20-pound bucket from the ground to the top of a 30-foot building with a cable that weighs three pounds per foot?
    (b) How much work is done lifting the same bucket from the ground to a height of 15 feet with the same cable?

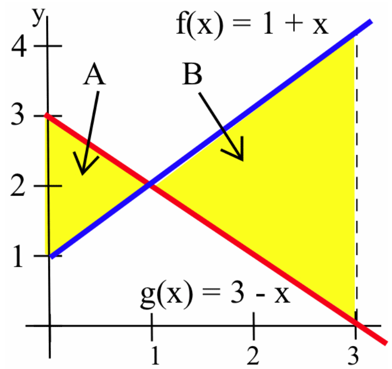
38. (a) How much work is done lifting a 60-pound chair from the ground to the top of a 20-foot building with a cable that weighs 1 pound per foot?
    (b) How much work is done lifting the same chair from the ground to a height of 5 feet with the same cable?

39. (a) How much work is done lifting a 10-pound calculus book from the ground to the top of a 30-foot building with a cable that weighs 2 pounds per foot?
    (b) From the ground to a height of 10 feet?
    (c) From a height of 10 feet to a height of 20 feet?

40. How much work is done lifting an 80-pound injured child to the top of a 20-foot hole using a stretcher weighing 14 pounds and a cable that weighs 1 pound per foot?

41. How much work is done lifting a 60-pound injured child to the top of a 15-foot hole using a stretcher weighing 10 pounds and a cable that weighs 2 pound per foot?

42. How much work is done lifting a 120-pound injured adult to the top of a 30-foot hole using a stretcher weighing 10 pounds and a cable that weighs 2 pound per foot?



4.7 Practice Answers

1. Referring to a graph (see margin figure) and using geometry: $A = \frac{1}{2}(2)(1) = 1$ and $B = \frac{1}{2}(4)(2) = 4$ so the total area is $1 + 4 = 5$.
   Referring to a graph of the functions and using integrals:
   $$\begin{aligned} A &= \int_0^1 [(3-x)-(1+x)]\,dx = \int_0^1 [2-2x]\,dx \\ &= \Big[2x - x^2\Big]_0^1 = [2-1]-[0-0] = 1 \\ B &= \int_1^3 [(1+x)-(3-x)]\,dx = \int_1^3 [2x-2]\,dx \\ &= \Big[x^2 - 2x\Big]_1^3 = [9-6]-[1-2)] = 4 \end{aligned}$$
   which also results in a total area of $1 + 4 = 5$.

2. Using the average value formula:
   $$\begin{aligned} \frac{1}{9-0}\int_0^9 \left[5+\sqrt{t}\right]\,dt &= \frac{1}{9}\int_0^9 \left[5+t^{\frac{1}{2}}\right]\,dt = \frac{1}{9}\left[5t + \frac{2}{3}t^{\frac{3}{2}}\right]_0^9 \\ &= \frac{1}{9}\left[\left(45 + \frac{2}{3}\cdot 27\right) - (0+0)\right] = \frac{45+18}{9} = 7 \end{aligned}$$
   so the average hourly production rate is 7 cars per hour.

3. $(\text{force})\cdot(\text{displacement}) = (10 \text{ pounds})\cdot(30 \text{ feet}) = 300 \text{ foot-pounds}$

4. (a) The work required to move the object a distance of 10 feet is $(10 \text{ pounds})\cdot(10 \text{ feet}) = 100$ foot-pounds. The work required to move the top 10 feet of the cable onto the roof is:
   $$\int_0^{10} (10-x)\cdot 3\,dx = \left[30x - \frac{3}{2}x^2\right]_0^{10} = [300-150]-[0] = 150 \text{ ft-lbs}$$
   and the force required to move the remaining 40 feet of cable is:
   $$(40 \text{ ft})\cdot\left(3\,\frac{\text{lbs}}{\text{ft}}\right)(10 \text{ ft}) = 1200 \text{ ft-lbs}$$
   so the total work required is $100 + 150 + 1200 = 1450$ foot-pounds.

   (b) The work required to move the object a distance of 10 feet is again $(10 \text{ pounds})\cdot(10 \text{ feet}) = 100$ foot-pounds. The work required to move the top 10 feet of the cable onto the roof is again 150 foot-pounds, and the force required to move the remaining 30 feet of cable is:
   $$(30 \text{ ft})\cdot\left(3\,\frac{\text{lbs}}{\text{ft}}\right)(10 \text{ ft}) = 900 \text{ ft-lbs}$$
   so the total work required is $100 + 150 + 900 = 1150$ foot-pounds.

4.8 Using Tables (and Technology) to Find Antiderivatives

Appendix I shows patterns for many antiderivatives — some of which you should already know based on your work in this chapter. Many reference books and Web sites contain far more than the ones listed in the appendix. A table of integrals helps you while you are learning calculus and serves as a reference later when you are using calculus.

Think of an integral table as a dictionary: something to use when you need to spell a challenging word or need the meaning of a new word. It would be difficult to write a report if you had to look up the spelling of *every* word, and it will be difficult to learn and use calculus if you have to look up every antiderivative. Tables of antiderivatives are limited by necessity and often take longer to use than finding an antiderivative from scratch, but they can also be very valuable and useful.

This section shows how to transform some integrals into forms found in Appendix I and how to use "recursion" formulas found in integral tables. The first Examples and Practice problems illustrate some of the techniques used to change an integral into a standard form.

These techniques are useful whether that standard form resides in a table or in your head.

Example 1. Use Appendix I to find $\int \frac{1}{9+x^2}\,dx$.

Solution. The integrand is a rational function, and the first entry you see listed in the "Rational Functions" section of Appendix I should be:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

This resembles the pattern we need, so replacing the a with 3 we have:

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{3^2+x^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

You can (and should) check this answer by differentiating. ◀

Appendix I (like some other integral tables) omits the "$+C$" arbitrary constant for conciseness, but you need to remember to include it when using the results of the table to find an indefinite integral.

Practice 1. Use Appendix I to find $\int \frac{1}{25+x^2}\,dx$ and $\int \frac{1}{25-x^2}\,dx$.

Notice that a small change in the form of the integrand (from $+$ to $-$ here) can lead to a very different result.

Example 2. Use Appendix I to find $\int \frac{1}{5+x^2}\,dx$.

Solution. The integrand is again a rational function, and the general form is the same as in the previous Example:

$$\int \frac{1}{5+x^2}\,dx = \int \frac{1}{(\sqrt{5})^2+x^2}\,dx = \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

but here we needed to put $a = \sqrt{5}$. ◀

The constant in the denominator of this integrand was not a perfect square, but the process is exactly the same — even if the result looks a bit "messier" due to the presence of the radical.

Practice 2. Use Appendix I to find $\int \frac{1}{7+x^2}\,dx$ and $\int \frac{1}{7-x^2}\,dx$.

We often need to perform some algebraic manipulations to change an integrand into one that exactly matches a pattern in the table.

Example 3. Use Appendix I to find $\int \frac{1}{9+4x^2}\,dx$.

Solution. The integrand is again a rational function, and the general form resembles the one used in the previous Examples, but here we have a $4x^2$ where we only see x^2 in the table pattern. To get the integrand in the form we want, we can factor a 4 out of the denominator:

$$\begin{aligned}\int \frac{1}{9+4x^2}\,dx &= \int \frac{1}{4\left(\frac{9}{4}+x^2\right)}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{3}{2}\right)^2+x^2}\,dx \\ &= \frac{1}{4}\cdot\frac{1}{\frac{3}{2}}\cdot\arctan\left(\frac{x}{\frac{3}{2}}\right)+C = \frac{1}{6}\arctan\left(\frac{2x}{3}\right)+C\end{aligned}$$

Another approach involves a change of variable. First write:

$$\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{3^2+(2x)^2}\,dx$$

We have $2x$ where we would like to see a standalone variable. To get that pattern, put $u = 2x \Rightarrow du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$:

$$\begin{aligned}\int \frac{1}{3^2+(2x)^2}\,dx &= \int \frac{1}{3^2+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right)+C \\ &= \frac{1}{6}\arctan\left(\frac{2x}{3}\right)+C\end{aligned}$$

which yields the same result as our previous method. ◀

Practice 3. Use Appendix I to find $\int \frac{1}{25+9x^2}\,dx$ and $\int \frac{1}{25-9x^2}\,dx$.

Sometimes a change of variable is absolutely necessary.

Example 4. Use Appendix I to find $\int \frac{e^x}{9+e^{2x}}\,dx$.

Solution. Here the integrand is *not* a rational function, but we can transform it into one by using the substitution $u = e^x \Rightarrow du = e^x\,dx$ so that $u^2 = (e^x)^2 = e^{2x}$:

$$\begin{aligned}\int \frac{e^x}{9+e^{2x}}\,dx &= \int \frac{1}{3^2+(e^x)^2}\cdot e^x\,dx = \int \frac{1}{3^2+u^2}\,du \\ &= \frac{1}{3}\arctan\left(\frac{u}{3}\right)+C = \frac{1}{3}\arctan\left(\frac{e^x}{3}\right)+C\end{aligned}$$

If you don't see the exact pattern you need in an integral table, try a substitution first. ◀

Practice 4. Evaluate $\int \frac{\cos(x)}{25+\sin^2(x)}\,dx$ and $\int \frac{\cos(x)}{25-\sin^2(x)}\,dx$.

How should you recognize whether algebra or a change of variable is needed? Experience and practice, practice, practice.

Using "Recursion" Formulas

A **recursion formula** gives one antiderivative in terms of another antiderivative. Usually the new antiderivative is somehow simpler than the original one. For example, the first recursion formula for a trigonometric function listed in Appendix I states:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

We will develop this formula from scratch in Problem 25 of Section 8.2. For now, you can check that it works by comparing the derivative of your answer to the original integrand for an integration problem that uses this — or any other — recursion formula.

This formula would allow us to write $\int \sin^8(x)\,dx$, for instance, in terms of $\int \sin^6(x)\,dx$, which should (theoretically, at least) be easier to compute than the original integral.

Example 5. Use a recursion formula to evaluate $\int \sin^4(x)\,dx$.

Solution. Applying the formula given in the discussion above:

$$\int \sin^4(x)\,dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx$$

This new integral is one we already know how to evaluate:

$$\int \sin^2(x)\,dx = \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + K$$

Putting this together with the result of the recursion formula, we get:

$$\begin{aligned}\int \sin^4(x)\,dx &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right] + C\\ &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{8}x - \frac{3}{16}\sin(2x) + C\end{aligned}$$

We could have included the "$+K$" here but then the result at the next stage would have included the constant terms

$$\cdots + \frac{3}{4}K + C$$

which is also an arbitrary constant.

We could have used Appendix I to find $\int \sin^2(x)\,dx$ instead — or even applied the recursion formula a second time to rewrite $\int \sin^2(x)\,dx$ in terms of $\int \sin^0(x)\,dx = \int 1\,dx$. ◀

Practice 5. Use Appendix I to evaluate $\int \cos^4(x)\,dx$ and $\int \cos^4(7x)\,dx$.

Using Technology

Many Web sites (such as Wolfram|Alpha, `www.wolframalpha.com`), computer programs (wxMaxima is a good free one) and calculators (such as the TI-89 or TI-Nspire CAS) feature computer algebra systems that can find antiderivatives of a wide variety of functions. For example, typing `integral sin^4(x)` into Wolfram|Alpha yields:

$$\int \sin^4(x)\,dx = \frac{1}{32}\,(12\,x - 8\sin(2\,x) + \sin(4\,x)) + \text{constant}$$

Although technology can help us find an antiderivative and evaluate a definite integral, in an application problem **you** still need to set up the Riemann sum that leads to the definite integral.

which (applying some trig idenities) agrees with our result above.

4.8 Problems

In Problems 1–48, use patterns and recursion formulas from the integral table in Appendix I as necessary (along with any other antiderivatives and integration techniques you already know) to evaluate each integral.

1. $\displaystyle\int \frac{1}{4+x^2}\,dx$

2. $\displaystyle\int \frac{5}{4+x^2}\,dx$

3. $\displaystyle\int \left[2x+\frac{2}{25+x^2}\right]\,dx$

4. $\displaystyle\int \frac{1}{4-x^2}\,dx$

5. $\displaystyle\int \frac{2}{9-x^2}\,dx$

6. $\displaystyle\int \left[\cos(x)+\frac{3}{25-x^2}\right]\,dx$

7. $\displaystyle\int \frac{1}{3+x^2}\,dx$

8. $\displaystyle\int \frac{5}{7+x^2}\,dx$

9. $\displaystyle\int \left[e^x+\frac{7}{2+x^2}\right]\,dx$

10. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

11. $\displaystyle\int \frac{3}{\sqrt{5-x^2}}\,dx$

12. $\displaystyle\int \frac{3}{\sqrt{4-x^2}}\,dx$

13. $\displaystyle\int \frac{1}{4+25x^2}\,dx$

14. $\displaystyle\int \frac{2}{\sqrt{9-16x^2}}\,dx$

15. $\displaystyle\int \frac{5}{\sqrt{1-4x^2}}\,dx$

16. $\displaystyle\int \sec(x+5)\,dx$

17. $\displaystyle\int \frac{2}{\sqrt{1+9x^2}}\,dx$

18. $\displaystyle\int x\cdot\sec(2x^2+7)\,dx$

19. $\displaystyle\int \ln(x+1)\,dx$

20. $\displaystyle\int \ln(3x-1)\,dx$

21. $\displaystyle\int 3x\cdot\ln(5x^2+7)\,dx$

22. $\displaystyle\int e^x\ln\left(e^x-3\right)\,dx$

23. $\displaystyle\int \cos(x)\cdot\ln\left(\sin(x)\right)\,dx$

24. $\displaystyle\int \frac{2}{\sqrt{x^2-9}}\,dx$

25. $\displaystyle\int \sqrt{4+x^2}\,dx$

26. $\displaystyle\int \sqrt{9+x^2}\,dx$

27. $\displaystyle\int \sqrt{16+x^2}\,dx$

28. $\displaystyle\int_0^1 \frac{1}{4+x^2}\,dx$

29. $\displaystyle\int_1^3 \left[2x+\frac{2}{25+x^2}\right]\,dx$

30. $\displaystyle\int_0^2 \frac{2}{9-x^2}\,dx$

31. $\displaystyle\int_{-1}^1 \frac{1}{3+x^2}\,dx$

32. $\displaystyle\int_0^1 \left[e^x+\frac{7}{2+x^2}\right]\,dx$

33. $\displaystyle\int_1^2 \frac{3}{\sqrt{5-x^2}}\,dx$

34. $\displaystyle\int_0^1 \frac{1}{4+25x^2}\,dx$

35. $\displaystyle\int_0^{0.1} \frac{5}{\sqrt{1-4x^2}}\,dx$

36. $\displaystyle\int_0^1 \frac{1}{\sqrt{9-4x^2}}\,dx$

37. $\displaystyle\int_0^6 \ln(x+1)\,dx$

38. $\displaystyle\int_0^3 3x\cdot\ln(5x^2+7)\,dx$

39. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\cdot\ln\left(2+\sin(x)\right)\,dx$

40. $\displaystyle\int_0^2 \sqrt{4+x^2}\,dx$

41. $\displaystyle\int_{-3}^3 \sqrt{9+x^2}\,dx$

42. $\displaystyle\int_0^1 \sqrt{16+x^2}\,dx$

43. $\displaystyle\int \sin^3(x)\,dx$

44. $\displaystyle\int \cos^3(x)\,dx$

45. $\displaystyle\int \cos^5(x)\,dx$

46. $\displaystyle\int \sec^5(x)\,dx$

47. $\displaystyle\int x^2\cos(x)\,dx$

48. $\displaystyle\int x^2\sin^5(x)\,dx$

49. Before doing any calculations, predict which you expect to be larger:
   - the average value of $\sin(x)$ on $[0, \pi]$
   - the average value of $\sin^2(x)$ on $[0, \pi]$

   Then calculate each average to see if your prediction was correct.
50. Find the area of the region bounded by the graph of $f(x) = \ln(x)$, the x-axis and the lines $x = 1$ and $x = C$ when $C = e$, 10, 100 and 200.
51. Find the average value of $f(x) = \ln(x)$ on the interval $1 \le x \le C$ when $C = e$, 10, 100, 200.
52. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

   (a) $\displaystyle\int_0^1 e^x\,dx$ (b) $\displaystyle\int_0^1 xe^x\,dx$ (c) $\displaystyle\int_0^1 x^2e^x\,dx$
53. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

   (a) $\displaystyle\int_1^2 e^x\,dx$ (b) $\displaystyle\int_1^2 xe^x\,dx$ (c) $\displaystyle\int_1^2 x^2e^x\,dx$
54. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

   (a) $\displaystyle\int_0^\pi \sin(x)\,dx$

   (b) $\displaystyle\int_0^\pi x\sin(x)\,dx$

   (c) $\displaystyle\int_0^\pi x^2\sin(x)\,dx$
55. Evaluate $\displaystyle\int_0^C \frac{2}{1+x^2}\,dx$ for $C = 1$, 10, 20 and 30. Before doing the calculation, estimate the value of the integral when $C = 40$.

4.8 Practice Answers

1. The integral $\displaystyle\int \frac{1}{25+x^2}\,dx$ resembles the pattern from Example 1:

$$\int \frac{1}{25+x^2}\,dx = \int \frac{1}{5^2+x^2}\,dx = \frac{1}{5}\arctan\left(\frac{x}{5}\right) + C$$

The integrand in $\displaystyle\int \frac{1}{25-x^2}\,dx$ is also a rational function, but we need a different pattern from Appendix I (see margin) with $a = 5$:

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|$$

$$\int \frac{1}{25-x^2}\,dx = \int \frac{1}{5^2-x^2}\,dx = \frac{1}{10}\ln\left|\frac{x+5}{x-5}\right| + C$$

2. The integral $\displaystyle\int \frac{1}{7+x^2}\,dx$ matches the pattern in Example 2:

$$\int \frac{1}{7+x^2}\,dx = \int \frac{1}{(\sqrt{7})^2+x^2}\,dx = \frac{1}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C$$

For $\displaystyle\int \frac{1}{7-x^2}\,dx$ we need the pattern in the margin with $a = \sqrt{7}$:

$$\int \frac{1}{7-x^2}\,dx = \int \frac{1}{(\sqrt{7})^2-x^2}\,dx = \frac{1}{2\sqrt{7}}\ln\left|\frac{x+\sqrt{7}}{x-\sqrt{7}}\right| + C$$

3. For the integral $\int \frac{1}{25+9x^2}\,dx$ we can factor 9 from the denominator:

$$\int \frac{1}{25+9x^2}\,dx = \int \frac{1}{9\left(\frac{25}{9}+x^2\right)}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{5}{3}\right)^2+x^2}\,dx$$

$$= \frac{1}{9}\cdot\frac{1}{\frac{5}{3}}\arctan\left(\frac{x}{\frac{5}{3}}\right)+C = \frac{1}{15}\arctan\left(\frac{3x}{5}\right)+C$$

and proceed as before. We could proceed similarly for $\int \frac{1}{25-9x^2}\,dx$ or we could substitute $u = 3x$ (see margin):

$u = 3x \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3}\,du$

$$\int \frac{1}{25-9x^2}\,dx = \int \frac{1}{5^2-(3x)^2}\,dx = \int \frac{1}{5^2-u^2}\cdot\frac{1}{3}\,du$$

$$= \frac{1}{3}\cdot\frac{1}{2\cdot 5}\ln\left|\frac{u+5}{u-5}\right|+C = \frac{1}{30}\cdot\ln\left|\frac{3x+5}{3x-5}\right|+C$$

4. For $\int \frac{\cos(x)}{25+\sin^2(x)}\,dx$, first use the substitution in the margin:

$u = \sin(x) \Rightarrow du = \cos(x)\,dx$

$$\int \frac{\cos(x)}{25+\sin^2(x)}\,dx = \int \frac{1}{25+u^2}\,du$$

followed by the result of the first part of Practice 1:

$$\int \frac{1}{25+u^2}\,du = \frac{1}{5}\arctan\left(\frac{u}{5}\right)+C = \frac{1}{5}\arctan\left(\frac{\sin(x)}{5}\right)+C$$

For $\int \frac{\cos(x)}{25-\sin^2(x)}\,dx$ use the same substitution, followed by the result from the second part of Practice 1:

$$\int \frac{1}{25-u^2}\,du = \frac{1}{10}\ln\left|\frac{u+5}{u-5}\right|+C = \frac{1}{10}\ln\left|\frac{\sin(x)+5}{\sin(x)-5}\right|+C$$

5. For $\int \cos^4(x)\,dx$ we need the recursion formula:

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x)+\frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

with $n = 4$:

$$\int \cos^4(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\int \cos^2(x)\,dx$$

$$= \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\int\left[\frac{1}{2}+\frac{1}{2}\cos(2x)\right]\,dx$$

$$= \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\left[\frac{1}{2}x+\frac{1}{4}\sin(2x)\right]+C$$

$u = 7x \Rightarrow du = 7\,dx \Rightarrow dx = \frac{1}{7}\,du$

$\int \cos^4(7x)\,dx = \frac{1}{7}\int \cos^4(u)\,du$

For $\int \cos^4(7x)\,dx$, first use a substitution (see margin) and then the result of the previous integration:

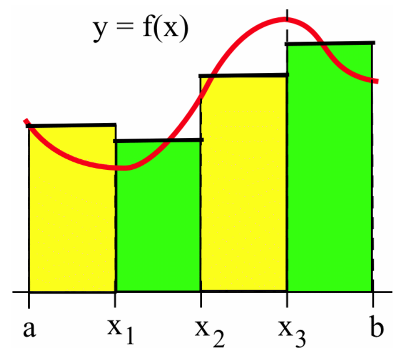
$$\int \cos^4(7x)\,dx = \frac{1}{28}\cos^3(7x)\sin(7x)+\frac{3}{28}\left[\frac{1}{2}(7x)+\frac{1}{4}\sin(14x)\right]+C$$

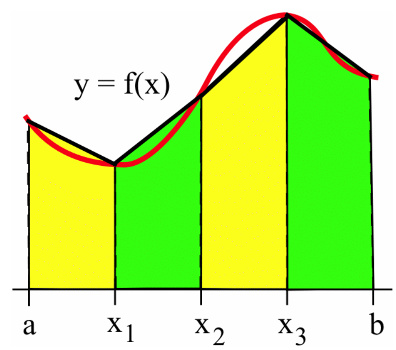
4.9 *Approximating Definite Integrals*

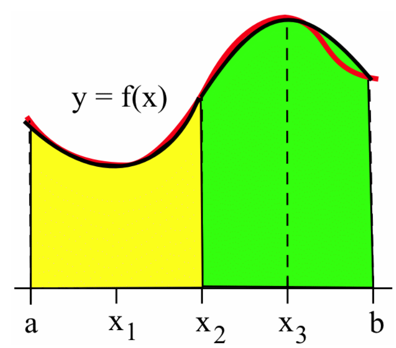
The Fundamental Theorem of Calculus tells how to calculate the exact value of a definite integral *if* the integrand is continuous and *if* we can find a formula for an antiderivative of the integrand. In practice, however, we may need to compute the definite integral of a function for which we only have table values or a graph — or of a function that does not have an elementary antiderivative. This section presents several techniques for getting approximate numerical values for definite integrals without using antiderivatives. Mathematically, exact answers are preferable and satisfying, but for most applications a numerical answer accurate to several digits is just as useful.

The ideas behind these methods are geometric and rather simple, but using the methods to get good approximations typically requires lots of arithmetic, something calculators and computers are very good (and quick) at doing.

The General Approach







The methods in this section approximate the definite integral of a function f by partitioning the interval of integration and building an "easy" function with values close to those of f on each interval, then evaluating the definite integrals of the "easy" functions exactly. If the "easy" functions are close to f, then the sum of the definite integrals of the "easy" functions should be close to the definite integral of f.

The **Left**, **Right** and **Midpoint Rules** approximate f with horizontal lines on each partition interval so the "easy" functions are constant functions, and the approximating regions are rectangles (see top margin figure). The **Trapezoidal Rule** approximates f with slanted lines, so the "easy" functions are linear and the approximating regions are trapezoids (see middle margin figure). Finally, **Simpson's Rule** approximates f with parabolas, so the "easy" functions are quadratic polynomials (see bottom margin figure).

The Left and Right approximation rules are simply Riemann sums with the point c_k in the k-th subinterval chosen to be the left or right endpoint of that subinterval. They typically require a large number of computations to get even mediocre approximations to the definite integral of f and are seldom used in practice. Along with the Midpoint Rule (which chooses each c_k to be the midpoint of the k-th subinterval), they are discussed near the end of the Problems for this section.

All of these methods partition the interval $[a, b]$ into n subintervals of equal width, so each subinterval has length $h = \Delta x_k = \dfrac{b-a}{n}$. The points of the partition are $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2 \cdot h$, $x_3 = a + 3 \cdot h$, and so on. The k-th point in the partition is given by the formula $x_k = a + k \cdot h$ and the last (n-th) point is thus:

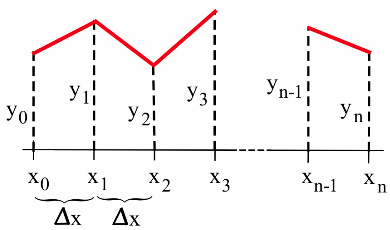
$$x_n = a + n \cdot h = a + n\left(\frac{b-a}{n}\right) = a + b - a = b$$

Approximating a Definite Integral Using Trapezoids

If the graph of f is curved, then slanted lines typically come closer to the graph of f than horizontal ones do. These slanted lines lead to trapezoidal approximating regions.

The area of a trapezoid is (base) · (average height) so the area of the first trapezoid in the margin figure is:

See Problem 29.



$$(\Delta x) \cdot \left(\frac{y_0 + y_1}{2}\right)$$

Similarly, the areas of the next few trapezoids are:

$$(\Delta x) \cdot \left(\frac{y_1 + y_2}{2}\right), \quad (\Delta x) \cdot \left(\frac{y_2 + y_3}{2}\right), \quad (\Delta x) \cdot \left(\frac{y_3 + y_4}{2}\right)$$

and so on, with the area of the last region being

$$(\Delta x) \cdot \left(\frac{y_{n-1} + y_n}{2}\right)$$

The sum of these n trapezoidal areas is:

$$\begin{aligned} T_n &= (\Delta x)\left(\frac{y_0 + y_1}{2}\right) + (\Delta x)\left(\frac{y_1 + y_2}{2}\right) + (\Delta x)\left(\frac{y_2 + y_3}{2}\right) + \cdots \\ &\qquad + (\Delta x)\left(\frac{y_{n-1} + y_n}{2}\right) \\ &= \left(\frac{\Delta x}{2}\right)\left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \cdots + (y_{n-1} + y_n)\right] \\ &= \left(\frac{h}{2}\right)\left[y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n\right] \\ &= \left(\frac{h}{2}\right)\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \end{aligned}$$

Each $f(x_k)$ value, except the first ($k = 0$) and the last ($k = n$), is the right-endpoint height of one trapezoid and the left-endpoint height of the next, so it shows up in the calculation for two trapezoids and is multiplied by 2 in the formula for the trapezoidal approximation.

Trapezoidal Approximation Rule

If	f is integrable on $[a, b]$ and $[a, b]$ is partitioned into n subintervals of width $h = \frac{b-a}{n}$
then	the Trapezoidal approximation of $\int_a^b f(x)\,dx$ is:

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Example 1. Compute T_4, the Trapezoidal approximation of $\int_1^3 f(x)\,dx$ for $n = 4$, with the values of f in the margin table.

x	$f(x)$
1.0	4.2
1.5	3.4
2.0	2.8
2.5	3.6
3.0	3.2

Solution. The step size is $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$ so:

$$T_4 = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$
$$= \frac{0.5}{2}\left[4.2 + 2(3.4) + 2(2.8) + 2(3.6) + (3.2)\right] = (0.25)(27) = 6.75$$

so we can say that $\int_1^3 f(x)\,dx \approx 6.75$. ◀

Let's see how well the Trapezoidal Rule approximates an integral whose value we can compute exactly:

$$\int_1^3 x^2\,dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}[27-1] = \frac{26}{3} \approx 8.6666667$$

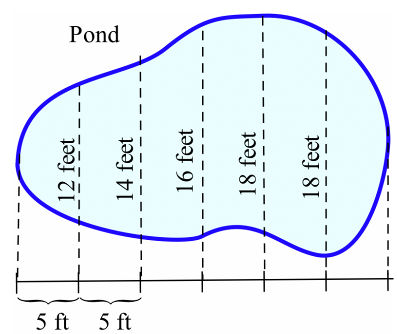
Example 2. Calculate T_4 for $\int_1^3 x^2\,dx$.

Solution. The step size is $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$ so:

$$T_4 = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$
$$= \frac{0.5}{2}\left[(1.0)^2 + 2(1.5)^2 + 2(2.0)^2 + 2(2.5)^2 + (3.0)^2\right]$$
$$= (0.25)\left[1 + 2(2.25) + 2(4) + 2(6.25) + 9\right] = 8.75$$

which is within 0.1 of the exact answer. Larger values for n give better approximations: $T_{20} = 8.67$ and $T_{100} = 8.6668$. ◀

Practice 1. On a summer day, the level of the pond shown in the margin fell 0.1 feet because of evaporation. Use the Trapezoidal Rule to approximate the surface area of the pond and then estimate how much water evaporated.



Approximating a Definite Integral Using Parabolas

If the graph of f is curved, the slanted lines from the Trapezoidal Rule may not fit the graph of f as closely as we would like, requiring a large number of subintervals to achieve a good approximation of the definite integral. Curves typically fit the graph of f better than straight lines in such situations, and the easiest nonlinear curves we know are parabolas.

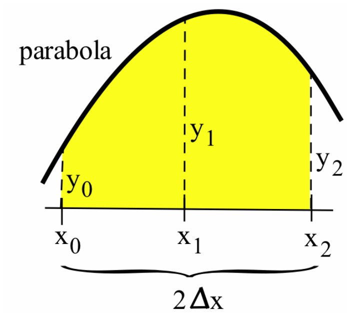
Just as we need two points to determine an equation of a line, we will need three points to determine an equation of a parabola.

This parabolic method is known as **Simpson's Rule**, named after British mathematician and inventor Thomas Simpson (1710–1761); Germans call it *Kepler'sche Fassregel*, after Johannes Kepler, who developed it 100 years before Simpson.

Calling these points (x_0, y_0), (x_1, y_1) and (x_2, y_2), the area under a parabolic region with evenly spaced x_k values (see margin) is:

This result is not obvious; see Problem 32 for the necessary algebra.

$$(2\Delta x) \cdot \left[\frac{y_0 + 4y_1 + y_2}{6}\right] = \frac{\Delta x}{3}\left[y_0 + 4y_1 + y_2\right]$$



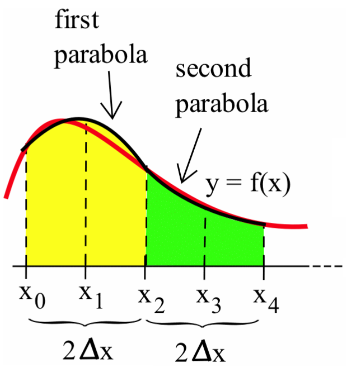
Taking the subintervals in pairs, the areas of the next few parabolic regions are:

$$\frac{\Delta x}{3}\left[y_2 + 4y_3 + y_4\right], \quad \frac{\Delta x}{3}\left[y_4 + 4y_5 + y_6\right], \quad \frac{\Delta x}{3}\left[y_6 + 4y_7 + y_8\right]$$

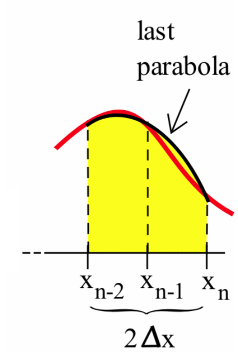
and so on, with the area of the last pair of regions being:

$$\frac{\Delta x}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

so the sum of all n parabolic areas (see margin) is:



$$\begin{aligned} S_n &= \frac{\Delta x}{3}\left[y_0 + 4y_1 + y_2\right] + \frac{\Delta x}{3}\left[y_2 + 4y_3 + y_4\right] + \cdots + \frac{\Delta x}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right] \\ &= \left(\frac{h}{3}\right)\left[(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 \cdots + y_{n-2} + 4y_{n-1} + y_n\right] \\ &= \left(\frac{h}{3}\right)\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-1} + 4y_{n-1} + y_n)\right] \\ &= \left(\frac{h}{3}\right)\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \right. \\ &\qquad \left. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right] \end{aligned}$$



In order to use **pairs** of subintervals, the number n of subintervals must be **even**. Notice that the coefficient pattern for the area under a single parabolic region is 1–4–1, but when we put several parabolas next to each other, they share some edges and the pattern becomes 1–4–2–4–2–$\cdots$–2–4–1 with the shared edges getting counted twice.

Parabolic Approximation Rule (Simpson's Rule)

If f is integrable on $[a, b]$ and $[a, b]$ is partitioned into n subintervals of length $h = \dfrac{b-a}{n}$

then the Parabolic approximation of $\displaystyle\int_a^b f(x)\,dx$ is:

$$S_n = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

x	$f(x)$
1.0	4.2
1.5	3.4
2.0	2.8
2.5	3.6
3.0	3.2

Example 3. Calculate S_4, the Simpson's Rule approximation of $\displaystyle\int_1^3 f(x)\,dx$ for the function f with values in the margin table.

Solution. The step size is $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$, so:

$$S_4 = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$
$$= \frac{\frac{1}{2}}{3}\left[4.2 + 4(3.4) + 2(2.8) + 4(3.6) + (3.2)\right] = \frac{1}{6}(41) = \frac{41}{6}$$

or approximately 6.833. ◀

Example 4. Calculate S_4 for $\int_1^3 2^x\,dx$.

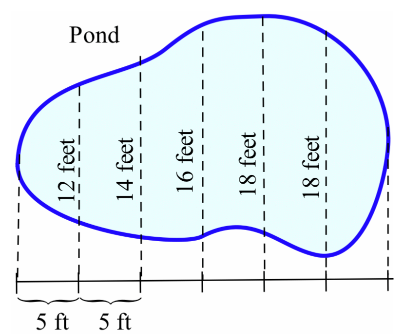
Solution. As in the previous Examples, $h = \frac{b-a}{n} = 0.5$ and $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$ and $x_4 = 3$.

$$S_4 = \frac{h}{3}\cdot\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$
$$= \frac{\frac{1}{2}}{3}\cdot\left[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\right]$$
$$= \left(\frac{1}{6}\right)\left[2^1 + 4\left(2^{1.5}\right) + 2\left(2^2\right) + 4\left(2^{2.5}\right) + \left(2^3\right)\right]$$
$$= \left(\frac{1}{6}\right)\left[2 + 4(2\sqrt{2}) + 2(4) + 4(4\sqrt{2}) + 8\right] = \left(\frac{1}{6}\right)\left[18 + 20\sqrt{2}\right]$$

or approximately 8.656854. The exact value of the integral is:

$$\int_1^3 2^x\,dx = \left[\frac{2^x}{\ln(2)}\right]_1^3 = \frac{8}{\ln(2)} - \frac{2}{\ln(2)} = \frac{6}{\ln(2)} \approx 8.65617024533$$

Larger values of n give better approximations: $S_{20} = 8.656171$ and $S_{100} = 8.656170$. ◀



Practice 2. Use Simpson's Rule to estimate the surface area of the pond in the margin figure.

Which Method Is Best?

The most difficult and time-consuming part of these approximations, whether done by hand or by computer, is the evaluation of the function at the x_k values. For n subintervals, all of the methods require about the same number of function evaluations. The table on the next page illustrates how closely each method approximates $\int_1^5 \frac{1}{x}\,dx = \ln(5) \approx 1.609437912$ using several values of n. The results in the table also show how quickly the actual error shrinks as the value of n increases: just doubling n from 4 to 8 cuts the actual error of the Simpson's Rule approximation of this definite integral by a factor of 9 — a good reward for our extra work.

The "error bounds" in the third column are discussed below.

method	**approximation**	**error bound**	**actual error**
T_4	1.683333333	0.6666666	0.07389542
S_4	1.622222222	0.5333333	0.01278431
L_4	2.083333333	2.0000000	0.47389542
R_4	1.283333333	2.0000000	0.32610458
M_4	1.574603175	0.3333333	0.03483474
T_8	1.628968254	0.1666666	0.01953034
S_8	1.610846561	0.0333333	0.00140865
L_8	1.828968254	1.0000000	0.21953034
R_8	1.428968254	1.0000000	0.18046966
M_8	1.599844394	0.0833333	0.00959352
T_{20}	1.612624844	0.0266667	0.00318693
S_{20}	1.609486789	0.0008533	0.00004888
L_{20}	1.692624844	0.4000000	0.08318693
R_{20}	1.532624844	0.4000000	0.07681307
M_{20}	1.607849324	0.0133333	0.00158859

Notice that for each value of n, the Simpson's Rule approximation S_n has the smallest error, and that the error for the Midpoint Rule approximation M_n (discussed in the Problems) is roughly half the error for the Trapezoidal Rule T_n. L_n and R_n denote the Left and Right approximations, respectively.

How Good Are the Approximations?

The approximation rules are valuable by themselves, but they are particularly useful because we can find "error bound" formulas that guarantee how close these approximations come to the exact values of the definite integral. It is useful to know that the value of an integral is "about 3.7," but we can have more confidence in our approximation if we know that value is "within 0.0001 of 3.7." Then we can decide if our approximation is good enough for the job at hand or if we need to improve it.

We can also solve the formulas for the error bounds provided below to determine how many subintervals we need to guarantee that our approximation is within some specified distance of the exact answer. There is no reason to use 1000 subintervals if 18 will give the needed accuracy. Unfortunately, the formulas for the error bounds require information about the derivatives of the integrands, so we cannot use these error bound formulas for the approximations of integrals of functions defined only by tables or graphs—or of continuous (hence integrable) functions that fail to have continuous derivatives.

The "error bound" formula for the Trapezoidal Rule approximation given at the top of the next page is just a "guarantee": the actual error is guaranteed to be no larger than the error bound. In fact, the actual error is usually much smaller than the error bound (compare the error bounds with the actual error for T_4, T_8 and T_{20} in the table above to see this principle in action).

The word "error" does not indicate a mistake, it simply means the deviation or distance of the approximate answers from the exact answer.

Error Bound for Trapezoidal Approximation

If f'' is continuous on $[a,b]$ and $|f''(x)| \le B_2$

then the "error" of the T_n approximation of $\int_a^b f(x)\,dx$ satisfies:

$$|\text{"error"}| = \left|\int_a^b f(x)\,dx - T_n\right| \le \frac{(b-a)^3 B_2}{12n^2}$$

While it's possible to prove this error bound formula using mathematics you've already learned, the proof is highly technical and sheds little or no insight into the workings of the Trapezoidal Rule, so we (like the authors of most calculus books) have omitted it.

Example 5. You can be certain that the T_{10} approximation of $\int_0^2 \sin(x^2)\,dx$ is within what distance of the exact value of the integral?

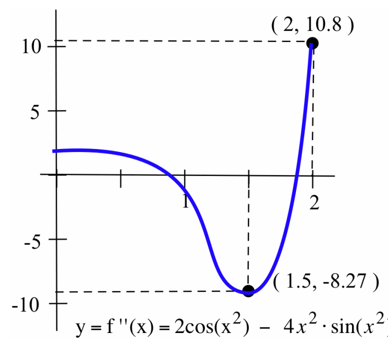
Solution. We know that $b-a=2$, $n=10$ and $f(x)=\sin\left(x^2\right)$, so $f''(x) = -4x^2\cdot\sin\left(x^2\right)+2\cdot\cos\left(x^2\right)$ is continuous on $[0,2]$.

Practice your differentiation skills by verifying this.

We now need an "upper bound" for $|f''(x)|$. If $f''(x)$ is differentiable (it is here) then we could use the techniques of Chapter 3 to find its maximum value on $[0,2]$ but that would require finding a *third* derivative of f, as well as some challenging algebra. Using the triangle inequality and the facts that $-1\le\sin(\theta)\le 1$ and $-1\le\cos(\theta)\le 1$, we can conclude:

$$|f''(x)| = \left|-4x^2\cdot\sin\left(x^2\right)+2\cdot\cos\left(x^2\right)\right| \le 4\cdot 2^2\cdot 1+2\cdot 1 = 18$$

so we could take $B_2=18$. We can do a bit better, however, by consulting a graph of $f''(x)$ on $[0,2]$ (see margin); it appears clear from the graph that $|f''(x)|\le 11$, so we take $B_2=11$ instead.



Using these values for a, b, n and B_2 in the "error bound" formula:

$$|\text{"error"}| = \left|\int_0^2 \sin(x^2)\,dx - T_{10}\right| \le \frac{2^3\cdot 11}{12\cdot 10^2} = \frac{88}{1200} = \frac{11}{150} < 0.074$$

so we can be certain that our T_{10} approximation of the definite integral is within 0.074 of the exact value:

$$T_{10} - 0.074 \le \int_0^2 \sin(x^2)\,dx \le T_{10} + 0.074$$

Computing $T_{10} = 0.7959247$, we can be certain that the value of the integral $\int_0^2 \sin(x^2)\,dx$ is somewhere between 0.722 and 0.870. ◀

Notice that (a bound for) the "error" depends on three things: the size of the interval of integration (the bigger the interval, the bigger the potential error); the number of subintervals in the partition (the more subintervals, the smaller the potential error); and the size of the second derivative of the integrand. We've already seen that the second derivative of a function is related to the concavity of its graph — later on we will learn that the second derivative helps measure the "curvature" of the graph of f; it should make sense that the more "curvy" a function is, the less effective a linear approximation technique would be.

Practice 3. Find an error bound for the T_{12} approximation of $\int_2^5 \frac{1}{x}\,dx$.

Example 6. How large must n be to be certain that T_n is within 0.001 of $\int_0^2 \sin(x^2)\,dx$?

Solution. Here we know the "allowable error" of 0.001 and we must find n. From Example 5 we know that $b-a=2$ and $B_2=11$, so we want the error bound to be less than the allowable error of 0.001:

$$\frac{2^3 \cdot 11}{12 \cdot n^2} < 0.001 \quad \Rightarrow \quad \frac{12 \cdot n^2}{88} > 1000 \quad \Rightarrow \quad n^2 > \frac{88000}{12}$$
$$\Rightarrow \quad n > \sqrt{\frac{22000}{3}} \approx 85.6$$

Because n must be an integer, we can take $n=86$. Computing $T_{86} \approx 0.80465$, we can be certain that the exact value of the integral is between 0.80365 and 0.80565. ◀

As often happens, T_{86} is even closer than 0.001 to the exact value of the integral:

$$\left| T_{86} - \int_0^2 \sin\left(x^2\right)\,dx \right| \approx 0.00012$$

Practice 4. Determine how large n must be in order to ensure that T_n is within 0.001 of $\displaystyle\int_2^5 \frac{1}{x}\,dx$.

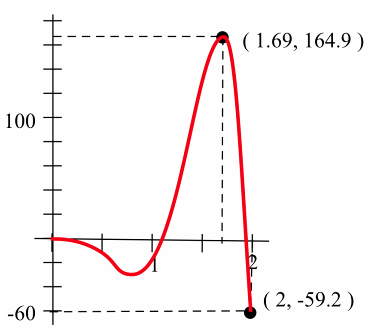
Error Bound for Simpson's Parabolic Approximation

If $f^{(4)}$ is continuous on $[a,b]$ and $\left|f^{(4)}(x)\right| \le B_4$

then the "error" of the S_n approximation of $\int_a^b f(x)\,dx$ satisfies:

$$|\text{“error”}| = \left| \int_a^b f(x)\,dx - S_n \right| \le \frac{(b-a)^5 B_4}{180 n^4}$$

Example 7. Find an error bound for the S_{10} approximation of $\displaystyle\int_0^2 \sin(x^2)\,dx$.



$f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$

Solution. We have $b-a=2$, $n=10$ and $f(x)=\sin(x^2)$, so $f^{(4)}(x) = (16x^4-12)\sin(x^2) - 48x^2\cos(x^2)$ is continuous on $[0,2]$. From a graph of $f^{(4)}(x)$ on $[0,2]$ (see margin), we can estimate that $B_4 = 165$, so

$$|\text{“error”}| = \left| \int_0^2 \sin(x^2)\,dx - S_{10} \right| \le \frac{2^5 \cdot 165}{180 \cdot 10^4} = \frac{5280}{1800000} < 0.003$$

and we can be certain that our S_{10} approximation of $\int_0^2 \sin(x^2)\,dx$ is within 0.003 of the exact value:

$$S_{10} - 0.003 \le \int_0^2 \sin(x^2)\,dx \le S_{10} + 0.003$$

Computing $S_{10} = 0.80537615$, we are certain that the exact value of $\int_0^2 \sin(x^2)\,dx$ is between 0.80237615 and 0.80837615. Notice that we achieved a much narrower guarantee using S_{10} compared to using T_{10} to approximate the same integral. ◀

Example 8. Determine how large n must be to ensure that S_n is within 0.001 of the exact value of $\displaystyle\int_0^2 \sin(x^2)\,dx$?

Solution. We want the "error bound" to be less than 0.001 and need to find n. We know that $b - a = 2$ and $B_4 = 165$

$$\frac{2^5 \cdot 165}{180 \cdot n^4} < 0.001 \quad \Rightarrow \quad \frac{180 \cdot n^2}{5280} > 1000 \quad \Rightarrow \quad n^2 > \frac{5280000}{180} = \frac{88000}{3}$$

$$\Rightarrow \quad n > \sqrt[4]{\frac{88000}{3}} \approx 13.09$$

Because n must be an even integer, we can take $n = 14$ and be certain that S_{14} is within 0.001 of $\int_0^2 \sin(x^2)\,dx$. ◀

As we have come to expect, S_{14} is even closer than 0.001 to the exact value of the integral; using advanced methods, we can show that:

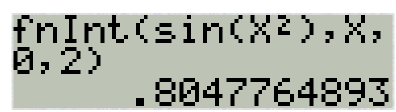
$$\left| \int_0^2 \sin(x^2)\,dx - S_{14} \right| \approx 0.00015$$

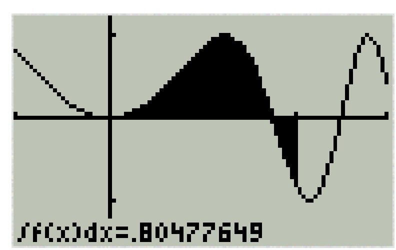
Alternative Methods

In Section 8.7 and in Chapter 10, you will learn how to approximate a function f over an entire interval $[a, b]$ using a single polynomial $p(x)$ of degree n; you can then approximate $\int_a^b f(x)\,dx$ with $\int_a^b p(x)\,dx$, which is relatively easy to compute. One advantage of this method is that (once we have found $p(x)$), we only need to evaluate another polynomial ($P(x)$ where $P'(x) = p(x)$) at two values ($P(a)$ and $P(b)$) to compute $\int_a^b p(x)\,dx \approx \int_a^b f(x)\,dx$ and we can get better approximations by increasing n and using polynomials of higher and higher degree; using the Trapezoidal Rule or Simpson's Rule requires us to evaluate $f(x)$ at $n+1$ points. A disadvantage of this approach is that our original $f(x)$ must have n continuous derivatives, which is not always the case, and we need to be able to compute those n derivatives at a single point. Most textbooks on Numerical Analysis offer more sophisticated techniques for approximating definite integrals.

Using Technology

If you have written even the most basic computer code, you should be able to write a program to compute any Trapezoidal Rule or Simpson's Rule approximation you want (accurate up to the floating-point limitations of the machine running your code). If you have a graphing calculator, it likely has one or more numerical integration utilities (see the margin for TI-83 output). The Web site Wolfram | Alpha (`www.wolframalpha.com`) can approximate definite integrals to any desired accuracy; typing `integral sin(x^2) from x=0 to x=2` yields:





$$\int_0^2 \sin(x^2)\,dx = \sqrt{\frac{\pi}{2}}\; S\left(2\sqrt{\frac{2}{\pi}}\right) \approx 0.804776$$

Wolfram | Alpha can also be used to quickly apply Simpson's Rule:

```
use Simpson's rule sin(x^2) from 0 to 2 with 10 intervals
```

yields an approximation of 0.804811 for $\int_0^2 \sin(x^2)\,dx$.

4.9 Problems

1. Use the values in the table below left to approximate $\int_2^6 f(x)\,dx$ by calculating T_4 and S_4.
2. Use the values in the table below left to approximate $\int_2^6 f(x)\,dx$ by calculating T_8 and S_8.

x	$f(x)$
2.0	2.1
2.5	2.7
3.0	3.8
3.5	2.3
4.0	0.3
4.5	-1.8
5.0	-0.9
5.5	0.5
6.0	2.2

x	$g(x)$
-3.0	4.2
-2.5	1.8
-2.0	0.7
-1.5	1.5
-1.0	3.4
-0.5	4.3
0.0	3.5
0.5	-0.3
1.0	-4.6

3. Use the values in the table above right to approximate $\int_{-3}^1 g(x)\,dx$ by calculating T_8 and S_8.
4. Use the values in the table above right to approximate $\int_{-3}^1 g(x)\,dx$ by calculating T_4 and S_4.

For Problems 5–10, calculate (a) T_4, (b) S_4 and (c) the exact value of the integral.

5. $\int_1^3 x\,dx$
6. $\int_0^2 [1-x]\,dx$
7. $\int_{-1}^1 x^2\,dx$
8. $\int_2^6 \frac{1}{x}\,dx$
9. $\int_0^\pi \sin(x)\,dx$
10. $\int_0^1 \sqrt{x}\,dx$

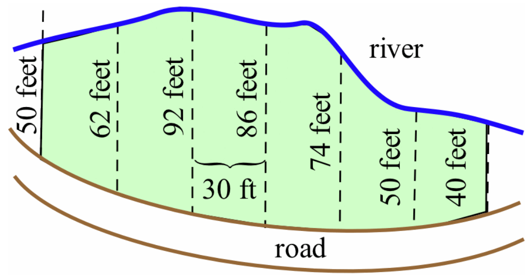
For Problems 11–16, calculate (a) T_6 and (b) S_6.

11. $\int_0^2 \frac{1}{1+x^3}\,dx$
12. $\int_1^2 2^x\,dx$
13. $\int_{-1}^1 \sqrt{4-x^2}\,dx$
14. $\int_0^1 e^{-x^2}\,dx$
15. $\int_1^4 \frac{\sin(x)}{x}\,dx$
16. $\int_0^1 \sqrt{1+\sin(x)}\,dx$

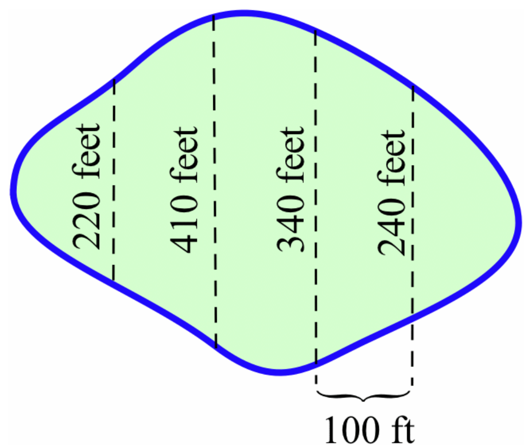
For 17–22, calculate (a) the error bound for T_4, (b) the error bound for S_4, (c) the value of n so that the error bound for T_n is less than 0.001, and (d) the value of n so that the error bound for S_n is less than 0.001.

17. $\int_1^3 x\,dx$
18. $\int_0^2 [1-x]\,dx$
19. $\int_{-1}^1 x^3\,dx$
20. $\int_2^6 \frac{1}{x}\,dx$
21. $\int_0^\pi \sin(x)\,dx$
22. $\int_0^1 \sqrt{x}\,dx$

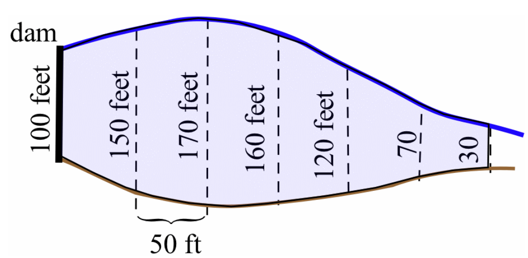
23. Estimate the area of the piece of land located between the river and the road in the figure below.



24. Estimate the area of the island in the figure below.



25. Estimate the volume of water in the reservoir shown below if the average depth is 22 feet.



26. The table below left shows the speedometer readings (in feet per minute) for a car at one-minute intervals. Estimate how far the car traveled (a) during the first 5 minutes of the trip and (b) during the first 10 minutes of the trip.

t	$v(t)$	t	$v(t)$
0	0	6	5200
1	2000	7	4400
2	3000	8	3000
3	5000	9	2000
4	5000	10	1200
5	6000		

t	$v(t)$	t	$v(t)$
0	0	6	520
1	420	7	440
2	540	8	360
3	300	9	260
4	500	10	180
5	580		

27. The table above right shows the speed (in feet per minute) of a jogger at one-minute intervals. Estimate how far the jogger ran during her workout.

28. Use the error-bound formula for Simpson's Rule to show that the parabolic approximation gives the exact value of $\int_a^b f(x)\,dx$ if $f(x) = Ax^3 + Bx^2 + Cx + D$ is a polynomial of degree 3 or less.

29. A trapezoidal region with base b and heights h_1 and h_2 (assume $h_1 \neq h_2$) can be cut into a rectangle with base b and height h_1 and a triangle with base b and height $h_1 - h_2$ (see figure at right). Show that the sum of the area of the rectangle and the area of the triangle is $b \cdot \left[\frac{h_1 + h_2}{2}\right]$.

30. Let $f(m)$ be the minimum value of f on the interval $[x_0, x_1]$, $f(M)$ be the maximum value of f on $[x_0, x_1]$, and $h = x_1 - x_0$. Show that:

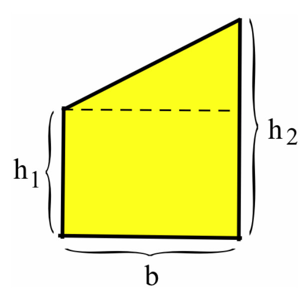
$$h \cdot f(m) \leq b \cdot \left[\frac{f(x_0) + f(x_1}{2}\right] \leq h \cdot f(M)$$

and use this result to show that the trapezoidal approximation is between the lower and upper Riemann sums for f. Because the limit (as $h \to 0$) of these Riemann sums is $\int_a^b f(x)\,dx$, conclude that the limit of the trapezoidal sums must equal $\int_a^b f(x)\,dx$.

31. Let $f(m)$ be the minimum value of f on the interval $[x_0, x_2]$, $f(M)$ the maximum of f on $[x_0, x_2]$ and $h = x_1 - x_0 = x_2 - x_1$. Show that the value

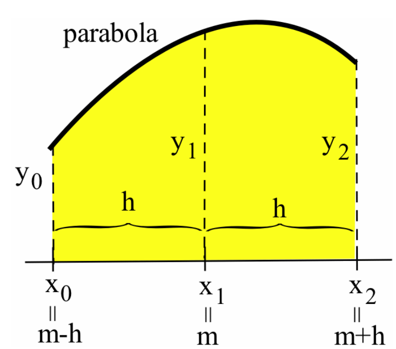
$$2h \cdot \left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right]$$

is between $2h \cdot f(m)$ and $2h \cdot f(M)$ and use this result to show that the parabolic approximation of $\int_a^b f(x)\,dx$ is between the lower and upper Riemann sums for f. Conclude that the limit of the parabolic sums must equal $\int_a^b f(x)\,dx$.



32. This problem guides you through the steps to show that the area under a parabolic region (see margin) with evenly spaced x_k values (which, for the purposes of this problem we will call $x_0 = m - h$, $x_1 = m$ and $x_2 = m + h$) is:

$$\frac{h}{3} \cdot [f(x_0) + 4f(x_1) + f(x_2)] = \frac{h}{3} \cdot [y_0 + 4y_1 + y_2]$$



(a) For $f(x) = Ax^2 + Bx + C$, verify that:

$$\int_{m-h}^{m+h} f(x)\,dx = \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big|_{m-h}^{m+h} = 2Am^2h + \frac{2}{3}Ah^3 + 2Bmh + 2Ch$$

(b) Expand each of the polynomials:

$$\begin{aligned} y_0 = f(m-h) &= A(m-h)^2 + B(m-h) + C \\ y_1 = f(m) &= Am^2 + Bm + C \\ y_2 = f(m+h) &= A(m+h)^2 + B(m+h) + C \end{aligned}$$

and use the results to verify that:

$$\begin{aligned} \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] &= 2h\left[\frac{f(m-h) + 4f(m) + f(m+h)}{6}\right] \\ &= 2Am^2h + \frac{2}{3}Ah^3 + 2Bmh + 2Ch \end{aligned}$$

(c) Compare the results of parts (a) and (b) to conclude that for any quadratic function $f(x) = Ax^2 + Bx + C$:

$$\int_{m-h}^{m+h} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Left-Endpoint, Right-Endpoint and Midpoint Rules

The rectangular approximation methods approximate an integrand with horizontal lines, so that the approximating regions are rectangles and the sum of the areas of these rectangular regions is a Riemann sum. The Left- and Right-Endpoint Rules are easy to understand and use, but they typically require a very large number of subintervals to ensure good approximations of a definite integral. The Midpoint Rule uses the value of the integrand at the midpoint of each subinterval: if these midpoint values of f are available (for example, when f is given by a formula) then the Midpoint Rule is often more efficient than the Trapezoidal rule. The rectangular approximation rules are:

$$\begin{aligned} L_n &= h \cdot \left[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right] \\ R_n &= h \cdot \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\right] \\ M_n &= h \cdot \left[f(c) + f(c+h) + f(c+2h) + \cdots + f(c+(n-1)h)\right] \end{aligned}$$

where $c = x_0 + \frac{h}{2}$ so that the points c, $c+h$, $c+2h$, etc. are the midpoints of the subintervals. The "error bounds" for these methods are:

$$\begin{aligned} |\text{“error” for } L_n \text{ or } R_n| &\le \frac{(b-a)^2 B_1}{2n} \\ |\text{“error” for } M_n| &\le \frac{(b-a)^3 B_2}{24n} \end{aligned}$$

where $B_1 \ge |f'(x)|$ on $[a,b]$ and $B_2 \ge |f''(x)|$ on $[a,b]$. Notice that the error bound for M_n is half the error bound of T_n, the trapezoidal approximation.

For Problems 33–38, calculate (a) L_4, (b) R_4, (c) M_4 and (d) the exact value of the integral.

33. $\int_1^3 x\,dx$

34. $\int_0^2 [1-x]\,dx$

35. $\int_{-1}^1 x^2\,dx$

36. $\int_2^6 \frac{1}{x}\,dx$

37. $\int_0^\pi \sin(x)\,dx$

38. $\int_0^1 \sqrt{x}\,dx$

39. Show that the Trapezoidal approximation is the average of the Left- and Right-Endpoint approximations: $T_n = \frac{1}{2}(L_n + R_n)$.

The integrals in Problems 40–43 will arise in applications from Chapter 5. Use technology to approximate each integral by applying Simpson's Rule with $n = 10$ and $n = 40$ to approximate their values. (Is S_{40} very different from S_{10}?)

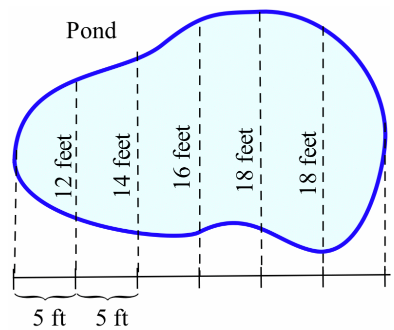
40. $\frac{1}{\sqrt{2\pi}} \int_{-2}^2 e^{-\frac{1}{2}x^2}\,dx$

41. $\int_{-1}^2 \sqrt{1+4x^2}\,dx$

42. $\int_0^\pi \sqrt{1+\cos^2(x)}\,dx$

43. $\int_0^{2\pi} \sqrt{16\sin^2(t) + 9\cos^2(t)}\,dt$

4.9 Practice Answers



1. Using the Trapezoidal Rule to approximate the pond's surface area:

$$T \approx \frac{5\text{ ft}}{2} \cdot [(0 + 2\cdot 12 + 2\cdot 14 + 2\cdot 16 + 2\cdot 18 + 2\cdot 18 + 0)\ \text{ft}] = 390\text{ ft}^2$$

so the volume is (surface area)(depth) $\approx \left(390\text{ ft}^2\right)(0.1\text{ ft}) = 39\text{ ft}^3$.

2. Using Simpson's Rule to approximate the pond's surface area:

$$S \approx \frac{5\text{ ft}}{3} \cdot [(0 + 4\cdot 12 + 2\cdot 14 + 4\cdot 16 + 2\cdot 18 + 4\cdot 18 + 0)\ \text{ft}] \approx 413\text{ ft}^2$$

3. $b - a = 3$, $n = 12$ and $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3}$, so on the interval $[2, 5]$:

$$|f''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{2^3} = \frac{1}{4}$$

We can therefore take $B_2 = \frac{1}{4}$, so:

$$|\text{error}| \le \frac{(b-a)^3 \cdot B_2}{12n^2} \le \frac{3^3 \cdot \frac{1}{4}}{12(12)^2} = \frac{27}{6912} \approx 0.004$$

4. We want:

$$|\text{error}| \le \frac{(b-a)^3 \cdot B_2}{12n^2} \le \frac{3^3 \cdot \frac{1}{4}}{12 \cdot n^2} = \frac{27}{48n^2} < 0.001$$

so solving for n:

$$\frac{48n^2}{27} > 1000 \Rightarrow n^2 > \frac{27000}{48} = \frac{1125}{2} \Rightarrow n > \sqrt{562.5} \approx 23.7$$

Using $n = 24$ will work. We can be certain that T_{24} is within 0.001 of the exact value of the integral. (We cannot guarantee that T_{23} is within 0.001 of the exact value of the integral, but it probably is.)